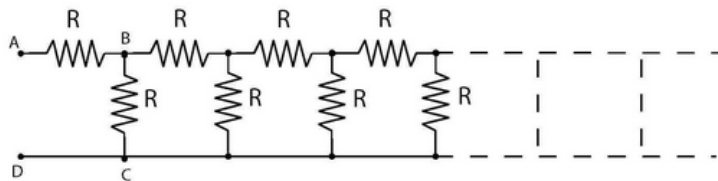
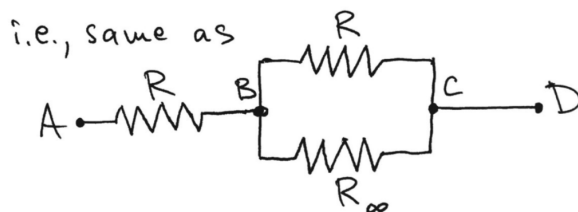
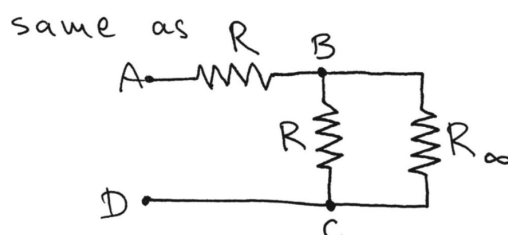
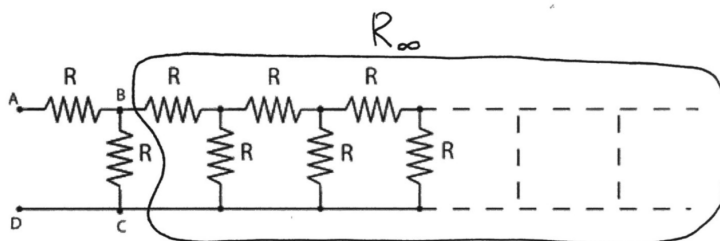


**Food for Thought Problem 1.<sup>1</sup> [Thinking simply]**

The total resistance of two resistors,  $R_1$  and  $R_2$ , in series is  $R_{\text{in series}} = R_1 + R_2$ , while their total resistance in parallel is given by  $\frac{1}{R_{\text{in parallel}}} = \frac{1}{R_1} + \frac{1}{R_2}$ . Use these facts to find the total resistance  $R_\infty$  between the points A and D of the infinite chain of resistors drawn in the figure below. The resistance of each resistor is  $R = 1 \text{ Ohm}$ .



You can find the total resistance  $R_\infty$  if you think similarly to Problem 1 from Homework 4. A simpler way to find  $R_\infty$  is given in the picture below.



This system of resistor is an example of a **self-similar system** – a system that looks the same as a part of it. Geometric sets with such property are called **fractals** – such sets occur quite often in Dynamical Systems.

<sup>1</sup>Food for Thought problems are NOT to be turned in, they are just for fun.

**Problem 2. [Poincaré-Bendixson Theorem]**

Consider the system

$$x' = x - y - x(x^2 + 5y^2) , \quad y' = x + y - y(x^2 + y^2) . \quad (1)$$

- (a) It is easy to see that the origin  $(0,0)$  of the  $(x,y)$ -plane is a fixed point. Linearize the system at  $(0,0)$  and classify the fixed point  $(0,0)$  (i.e., find out whether it is a saddle, stable spiral, unstable spiral, stable node, ...).
- (b) Rewrite the system in polar coordinates, using the identities  $rr' = xx' + yy'$  and  $\theta' = \frac{xy' - yx'}{r^2}$  (which are easy to obtain by differentiating the relations between Cartesian and polar coordinates, but you do not need to derive them here).

*Hint:* The answer is  $r' = r - r^3 - 4r^3 \cos^2 \theta \sin^2 \theta$ ,  $\theta' = 1 + 4r^2 \cos \theta \sin^3 \theta$ , but I want to see your calculations.

- (c) Prove that the maximum value of the function  $\varphi(\theta) := (\cos \theta \sin \theta)^2$  is  $\frac{1}{4}$  (if  $\theta$  is allowed to take any value). The easiest way to answer this question is to use some *very* elementary trigonometry. What is the minimum value that the function  $\varphi(\theta)$  takes?
- (d) Consider a circle of radius  $r_1$ , centered at the origin. Use your result from part (c) to show that, if  $r_1 < \frac{1}{\sqrt{2}}$ , then all trajectories have a radially outward component on it (i.e., that on this circle  $r' > 0$ ).
- (e) Determine the circle of minimum radius,  $r_2$ , centered at the origin such that all trajectories have a radially inward component on it.
- (f) Prove that the system (1) has a limit cycle in the trapping region  $r_1 \leq r \leq r_2$ . Figure 1 shows the result of numerical integration of the system (1). If you are taking the class as MATH 4193, you may assume without proof that there are no fixed points of the system (1) in the trapping region (this follows directly from part (g) which is only for those taking the class as 5103).

- (g) **Only if you take the class as 5103!**

Show that there are no fixed points in the trapping region found in part (f).

*Hint:* One way to prove this is to exclude  $r$  from the system  $\frac{r'}{r} = 0$ ,  $\theta' = 0$ , and then to show that the resulting equation for  $\theta$  has no solution. To avoid long calculations, you may use a computer to plot some function of one variable.

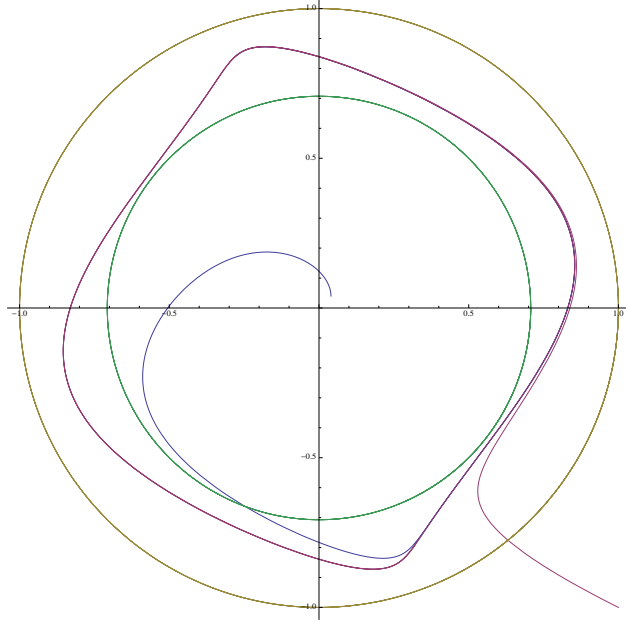


Figure 1: A limit cycle of the system (1), two phase trajectories, and the boundaries of the trapping region.

### Problem 3. [Nullclines and bifurcations]

In the article

P. Gray and S. Scott, Sustained oscillations and other exotic patterns of behavior in isothermal reactions, *Journal of Physical Chemistry*, Vol. 89 (1985), pp. 22–32

the authors consider a hypothetical isothermal autocatalytic reaction whose kinetics are given in dimensionless form by the two-parameter system

$$\begin{aligned} x' &= a(1 - x) - xy^2 , \\ y' &= xy^2 - (a + b)y . \end{aligned} \tag{2}$$

Here  $a > 0$  is a positive parameter and  $b$  is a parameter that can take any value. The functions  $x(t)$  and  $y(t)$  may take any values in  $\mathbb{R}$ .

(a) Show that the equations of the nullclines can be written as follows:

$$\begin{aligned} (x' = 0)\text{-nullcline} : \quad x &= \varphi(y) := \frac{a}{a + y^2} , \\ (y' = 0)\text{-nullcline} : \quad x &= \psi(y) := \frac{a + b}{y} \quad \text{or} \quad y \equiv 0 . \end{aligned} \tag{3}$$

- (b) Prove that at  $b = -a + \frac{1}{2}\sqrt{a}$ , the nullclines  $\{x = \varphi(y)\}$  and  $\{x = \psi(y)\}$  become tangent. Show that the coordinates  $(x^*, y^*)$  of the points where the tangency occurs are  $(\frac{1}{2}, \sqrt{a})$ . Figure 2 shows the nullclines for three different pairs of values  $(a, b)$ .

*Remark:* It is clear that the nullclines  $\{x = \varphi(y)\}$  and  $\{y \equiv 0\}$  can never be tangent, so you do not need to consider this.

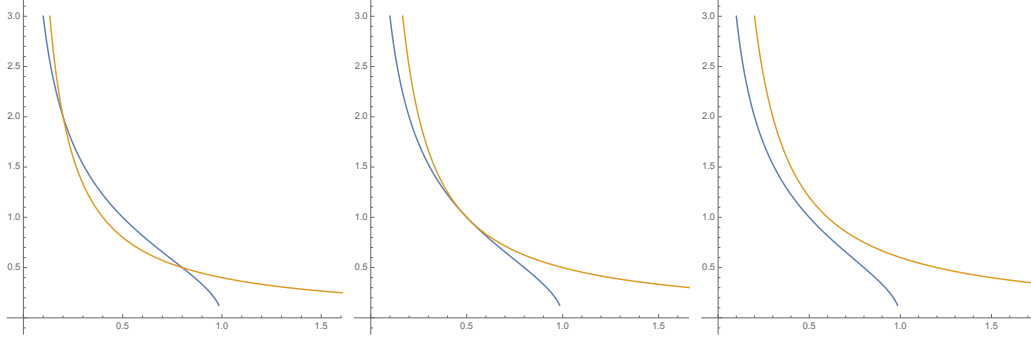


Figure 2: Nullclines of the nonlinear system (2) for  $(a, b) = (1, -0.6)$  (left);  $(a, b) = (1, -0.5)$  (center);  $(a, b) = (1, -0.4)$  (right).

- (c) What is the significance of what you found in part (b)? Discuss its meaning from point of view of the bifurcations occurring in the systems, naming specifically the type of bifurcations that occurs.
- (d) Let  $a = 1$ . We know from part (b) that when  $b = -1 + \frac{1}{2}\sqrt{1} = -\frac{1}{2}$ , the nullclines are tangent at the point  $(x^*, y^*) = (\frac{1}{2}, 1)$ . Let us consider what happens if  $b$  is slightly off, i.e., take  $b = -\frac{1}{2} + \xi$ , where  $\xi$  is a very small number (positive or negative). For these values of  $a$  and  $b$ , the equations (3) of the nullclines become  $x = \frac{1}{1+y^2}$ ,  $x = \frac{\frac{1}{2}+\xi}{y}$ . Show that from this system we can obtain the following quadratic equation for the  $y$ -coordinate  $y^*$  of the fixed points:

$$(y^*)^2 - (\tfrac{1}{2} + \xi)^{-1} y^* + 1 = 0 .$$

Since  $\xi$  is very small, we can use the formula for the geometric series to easily obtain the Taylor expansion of  $\frac{1}{\frac{1}{2}+\xi}$  at  $\xi = 0$ :

$$\frac{1}{\frac{1}{2} + \xi} = \frac{2}{1 - (-2\xi)} = 2 [1 + (-2\xi) + (-2\xi)^2 + \dots] \approx 2(1 - 2\xi) ,$$

so in this approximation the quadratic equation for  $y^*$  becomes

$$(y^*)^2 - 2(1 - 2\xi) y^* + 1 = 0 .$$

Show that, neglecting all high-order in  $\xi$  terms, we can write the solutions of this quadratic equation as  $y_{1,2}^* = 1 - 2\xi \pm 2\sqrt{-\xi}$ . From this expression for  $y_{1,2}^*$ , we see that there are no solutions for  $\xi > 0$  while there are two solutions for  $\xi < 0$ . What is the reason for this?

- (e) Leaving only the term of lowest order with respect to  $\xi$ , we can write  $y_{1,2}^* \approx 1 \pm 2\sqrt{-\xi}$ . In the same approximation, find the corresponding values  $x_{1,2}^*$ . Please write your calculations in detail. Figure 3 shows the stream plots of the system for two different pairs  $(a, b)$ , for one of which there are fixed points in the  $(x, y)$ -plane, while for the others there are no fixed points. The exact coordinates of the fixed points are given for comparison with their approximate values obtained in parts (d) and (e).

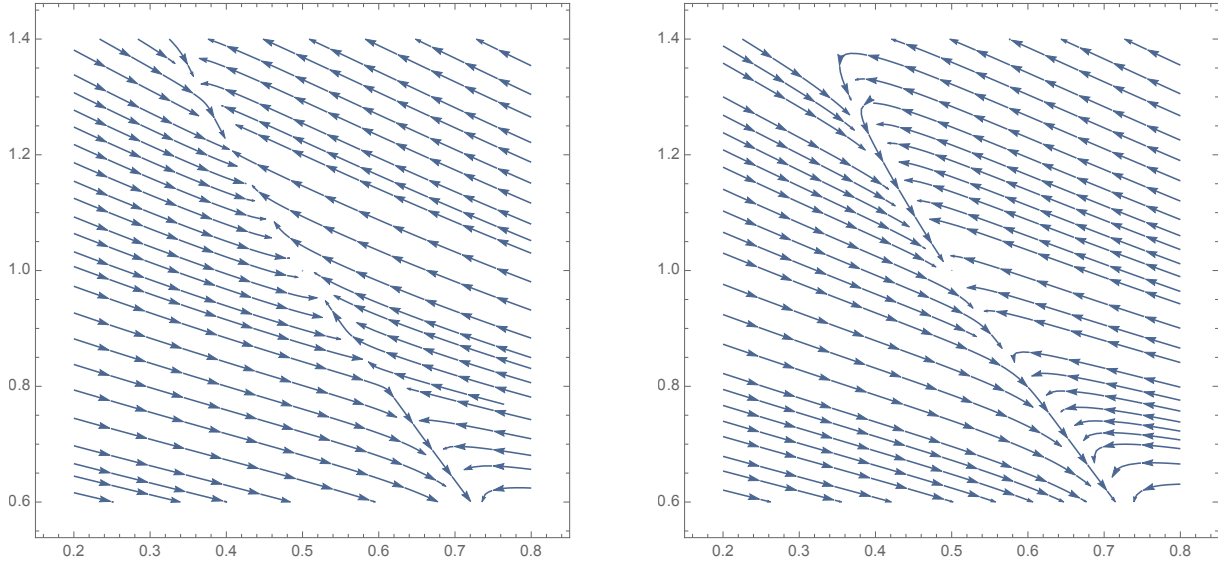


Figure 3: Integral lines of the nonlinear system (2) for  $(a, b) = (1, -0.51)$  (left) and  $(a, b) = (1, -0.49)$  (right). There are no fixed points in the figure on the right. The exact coordinates of the fixed points in the figure on the left are  $(x_1^*, y_1^*) = (0.4005012563 \dots, 1.223466824 \dots)$  and  $(x_1^*, y_2^*) = (0.5994987437 \dots, 0.8173495026 \dots)$ , while in the approximation used in parts (d) and (e),  $(x_1^*, y_1^*) \approx (0.4, 1.22)$  and  $(x_2^*, y_2^*) \approx (0.6, 0.82 \dots)$ .