MATH 4163 **Homework 5** **Due Tue, 02/27/2018**

Problem 1. [Dealing with non-zero boundary conditions]
Consider the initial-boundary value problem (IBVP) for the diffusion equation for the function $u(x,t)$ on the spatial interval $[0,L]$, with non-zero Dirichlet BCs:

$$\begin{aligned} & u_t = \alpha^2 u_{xx} \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\ & u(0,t) = a \ , \\ & u(L,t) = b \ , \\ & u(x,0) = f(x) \ , \end{aligned}$$

where a and b are constants. In class we learned how to solve this IBVP in the case $a = 0$, $b = 0$. The IBVP with $a \neq 0$, $b \neq 0$ can easily be reduced to an IBVP with homogeneous (i.e., zero) BCs by the simple trick outlined below.

(a) Let $g(x) = \beta x + \gamma$ be a linear function on $[0,L]$. Choose values of the constants β and γ such that

$$g(0) = a \ , \quad g(L) = b \ .$$

(b) Introduce a new function $v(x,t)$ by

$$u(x,t) =: v(x,t) + g(x) \ ,$$

where g is the function obtained in part (a). Derive the IBVP for the function v coming form the IBVP for u. Write clearly the PDE, BCs and IC satisfied by v.

Problem 2. [Separation of variables in the heat equation in $\mathbb{R}^2$ in a special case]
The heat equation in $\mathbb{R}^n$ has the form $u_t = \alpha^2 \Delta u$, where $u = u(\mathbf{x},t)$, $\alpha = \text{const} > 0$, and

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the Laplacian. In $\mathbb{R}^2$, if one uses polar coordinates (r,θ), it can easily be shown that

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \ .$$

Consider the heat equation in $\mathbb{R}^2$ and assume that the function u depends only on the radial coordinate and on the time, i.e., $u = u(r,t)$. In this case the heat equation reads

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) \ .$$

Look for a solution of this equation of the form $u(r,t) = R(r)T(t)$ and derive ordinary differential equations for the functions R and T. Do not forget that there should be a constant from the separation of variables (in class we called it μ). Do *not* attempt to solve the equations you obtained.

Problem 3. [Equivalence of norms on $\mathbb{R}^n$]

Two norms, $\| \ \|$ and $\| \ \|'$, on the same vector space V are said to be *equivalent* if there exist positive constants C_1 and C_2 such that

$$C_1\|\mathbf{u}\| \leq \|\mathbf{u}\|' \leq C_2\|\mathbf{u}\| \qquad \text{for any } \mathbf{u} \in V \ .$$

Consider the vector space $\mathbb{R}^n$ with the following norms defined on it:

$$\|\mathbf{u}\|_1 := \sum_{j=1}^{n} |u_j| \ , \qquad \|\mathbf{u}\|_2 := \left(\sum_{j=1}^{n} |u_j|^2 \right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty := \max_{1\leq j\leq n} |u_j| \ .$$

(a) Prove that the norms $\| \ \|_1$ and $\| \ \|_\infty$ on $\mathbb{R}^n$ are equivalent.

(b) Directly from the definition of equivalence of norms, prove that if the norms $\| \ \|$ and $\| \ \|'$ are equivalent and the norms $\| \ \|'$ and $\| \ \|''$ are equivalent, then the norms $\| \ \|$ and $\| \ \|''$ are equivalent.

Problem 4. [Unit disks in $\mathbb{R}^2$ endowed with different norms]

The *unit ball* (or the *unit disk* in case of two dimensions) in a normed linear space $(V, \| \ \|)$ is defined as the set of vectors $\mathbf{u}$ satisfying $\|\mathbf{u}\| \leq 1$.

(a) Draw the unit disk in $\mathbb{R}^2$ endowed with the norm $\| \ \|_2$. Explain briefly with an equation, i.e., write $\mathbf{u} = (u_1, u_2)$, so the vectors from the unit disk would satisfy $\|\mathbf{u}\|_2 = \|(u_1,u_2)\|_2 = \sqrt{|u_1|^2 + |u_2|^2} = 1$, and justify your drawing.

(b) Draw the unit disk in $\mathbb{R}^2$ endowed with the norm $\| \ \|_1$. Explain briefly with an equation.

(c) Draw the unit disk in $\mathbb{R}^2$ endowed with the norm $\| \ \|_\infty$. Explain briefly with an equation.

Problem 5. [Bases and projections in a space of polynomials]

In this problem, a "polynomial" means a polynomial P of a real variable with real coefficients (so that both x and $P(x)$ are real numbers). As discussed in class, the polynomials of order no higher than n form a linear space with respect to the addition of polynomials and multiplication of a polynomial by a real number as follows: if P and Q are polynomials of degree $\leq n$ and $\alpha \in \mathbb{R}$, then the polynomials $P + Q$ and αP are defined as

$$(P + Q)(x) := P(x) + Q(x) \ , \qquad (\alpha P)(x) := \alpha\, P(x) \ .$$

Let $V_n(a, b; w(x))$ stand for the linear space of polynomials defined on the interval with left end a and right end b (at each end, the interval can be open or closed; a and b can be finite or infinite) of degree no greater than n endowed with the inner product with weight function w:

$$\langle P, Q \rangle = \int_a^b P(x)\, Q(x)\, w(x)\, \mathrm{d}x \ .$$

Sanele defined a family of polynomials which he denoted (very modestly!) by S_0, S_1, S_2, ... These polynomials satisfy the following conditions:

(i) the polynomial S_k is of degree k;

(ii) the coefficient of x^k in S_k is equal to 1 (such polynomials are called *monic*);

(iii) the polynomials S_0, S_1, S_2, ..., S_n form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$ on $[0, \infty)$ of degree $\leq n$, with weight function $w(x) = \mathrm{e}^{-x}$.

In the solution of this problem the following identity will be handy (by definition, $0! = 1$):

$$\int_0^\infty x^k\, \mathrm{e}^{-x}\, \mathrm{d}x = k! \ , \qquad k = 0, 1, 2, \dots \ .$$

(a) Clearly, $S_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial S_1 of degree 1 that is orthogonal to S_0 (i.e., such that $\langle S_1, S_0 \rangle = 0$).
Hint: Set $S_1(x) = x + \alpha$ and compute the value of α for which $\langle S_1, S_0 \rangle = 0$.

(b) Find the only monic quadratic polynomial $S_2(x) = x^2 + \beta x + \gamma$ that is orthogonal to both S_0 and S_1.

(c) Show that the polynomial $P(x) = x^2 + 3$ can be represented as a linear combination of the polynomials S_0, S_1 and S_2 as follows: $P = S_2 + 4S_1 + 5S_0$.

(d) Show by direct integration that $\langle S_0, S_0 \rangle = 1$, $\langle S_1, S_1 \rangle = 1$, $\langle S_2, S_2 \rangle = 4$.

(e) Find the orthogonal projection, $\mathrm{proj}_{S_0+2S_1} P$, of the polynomial $P(x) = x^2 + 3$ onto the "straight line"

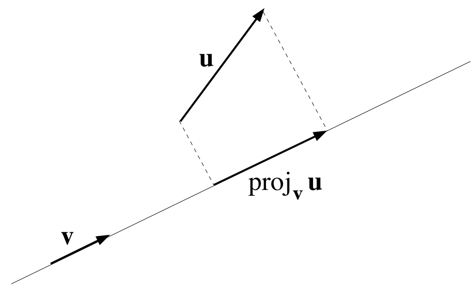
$$\ell := \{t(S_0 + 2S_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; \mathrm{e}^{-x})$. If you have solved parts (c) and (d), then finding this orthogonal projection should be easy.

Hint: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space V, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\, \mathbf{v}$$

– see the picture below.



(f) Finally, construct an orthonormal basis $\widetilde{S}_k$ ($k = 0, 1, 2$) in $V_2(0, \infty; \mathrm{e}^{-x})$ by setting $\widetilde{S}_k := \mu_k S_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\big\|\widetilde{S}_k\big\| := \sqrt{\big\langle \widetilde{S}_k, \widetilde{S}_k \big\rangle}\ ,$$

of the polynomial $\widetilde{S}_k$ is 1. Find the explicit expressions for $\widetilde{S}_0(x)$, $\widetilde{S}_1(x)$, and $\widetilde{S}_2(x)$.

Problem 6. [Antisymmetry of the differentiation operator]

An operator $A : V \to V$ in an inner product vector space $(V, \langle\ \rangle)$ is said to be *antisymmetric* if

$$\langle \mathbf{u}, A\mathbf{v} \rangle = -\langle \mathbf{v}, A\mathbf{u} \rangle \qquad \text{for any } \mathbf{u}, \mathbf{v} \in V\ .$$

Let $\mathcal{F}$ be the vector space of smooth functions $f : [0, L] \to \mathbb{R}$ that vanish at the endpoints:

$$\mathcal{F} := \{ f \in C^\infty([0, L])\ :\ f(0) = 0,\ f(L) = 0 \}\ .$$

Endow $\mathcal{F}$ with the inner product

$$\langle f, g \rangle := \int_0^L f(x)\, g(x)\, \mathrm{d}x\ , \qquad f, g \in \mathcal{F}\ .$$

Let $A := \frac{\mathrm{d}}{\mathrm{d}x} : \mathcal{F} \to \mathcal{F}$ be the differentiation operator: if $f \in \mathcal{F}$, then

$$(Af)(x) := f'(x)\ .$$

As you know from elementary Calculus, $(f + \alpha g)' = f' + \alpha g'$, so A is a linear operator.

Use integration by parts to prove that A is an antisymmetric operator on $\mathcal{F}$, i.e., show that $\langle f, Ag \rangle = -\langle g, Af \rangle$ for any $f, g \in \mathcal{F}$.

Problem 7. [Symmetry of the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$]

An operator $S : V \to V$ in an inner product vector space $(V, \langle\ \rangle)$ is said to be *symmetric* if

$$\langle \mathbf{u}, S\mathbf{v} \rangle = \langle \mathbf{v}, S\mathbf{u} \rangle \qquad \text{for any } \mathbf{u}, \mathbf{v} \in V\ .$$

Let $\mathcal{F}$ be the vector space of smooth functions $f : [0, L] \to \mathbb{R}$ that vanish at the left endpoint and whose first derivative vanishes at the right endpoint:

$$\mathcal{F} := \{ f \in C^\infty([0, L])\ :\ f(0) = 0,\ f'(L) = 0 \}\ .$$

Endow $\mathcal{F}$ with the inner product

$$\langle f, g \rangle := \int_0^L f(x)\, g(x)\, \mathrm{d}x \;, \qquad f, g \in \mathcal{F} \;.$$

Let $S := \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ be the second derivative (which, as you know, is a linear operator):

$$(Sf)(x) := f''(x) \;.$$

(a) Use integration by parts twice to show that, for any two smooth functions f and g defined on $[0, L]$, the following identity holds:

$$\int_0^L f g''\, \mathrm{d}x = (f g')\Big|_{x=0}^{L} - (f' g)\Big|_{x=0}^{L} + \int_0^L f'' g\, \mathrm{d}x$$

or, written in more detail,

$$\int_0^L f(x) g''(x)\, \mathrm{d}x = f(L)g'(L) - f(0)g'(0) - [f'(L)g(L) - f'(0)g(0)] + \int_0^L f''(x) g(x)\, \mathrm{d}x \;.$$

This identity is sometimes called *Green's formula.*

(b) Use the identity from part (a) to prove that S is a symmetric operator on $\mathcal{F}$, i.e., that $\langle f, Sg \rangle = \langle g, Sf \rangle$ for any $f, g \in \mathcal{F}$.