MATH 4073 Homework 5 Due Thu, 9/26/13

Problem 1. A thin wooden rod is attached to the point with coordinates $(0, 1)$ in the (x, y)-plane, and it is clamped at this point, so that it starts off in positive x-direction. The other end of the rod passes under a thin peg at the point $(10, 0)$. The rod is shown in Figure 1. Let the function

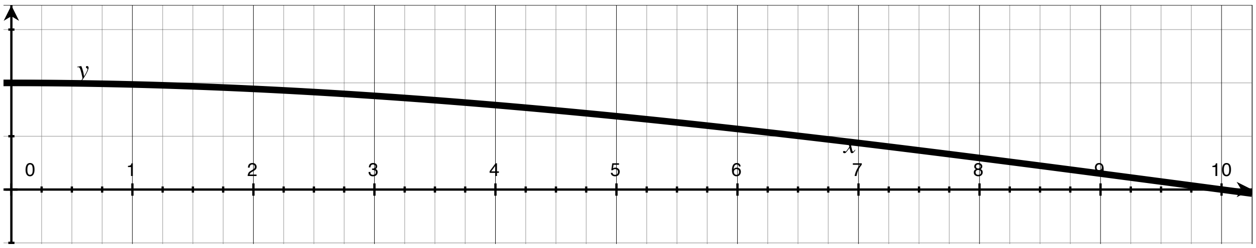

Figure 1: Shape of the wooden rod (see text).

$f(x)$ give the shape of the rod: $y = f(x)$. One can show that the bending of a rod is governed by the fourth order ordinary differential equation

$$f^{(4)}(x) = 0 \ . \tag{1}$$

According to the description above, the function $f(x)$ must satisfy the following boundary conditions:

$$\begin{aligned} f(0) &= 1 \quad \text{[the rod passes through the point } (0,1)] \ , \\ f(10) &= 0 \quad \text{[the rod passes through the point } (10,0)] \ , \\ f'(0) &= 0 \quad \text{[the rod is clamped at } (0,1) \text{ and "starts off" horizontally]} \ , \\ f''(10) &= 0 \quad \text{[the right end of the rod is free (no bending forces)]} \ . \end{aligned}$$

(a) Write down the general solution of the differential equation (1). Since this is a fourth order differential equation, its general solution must contain four arbitrary constants C_1, C_2, C_3, C_4.
Remark: The answer is obvious – you do not need to know how to solve differential equations in order to solve this part of the problem.

(b) Impose the boundary conditions to find the solution of the boundary value problem

$$f^{(4)}(x) = 0 \ , \quad f(0) = 1 \ , \quad f(10) = f'(0) = f''(10) = 0 \ .$$

Hint: I found that the coefficient in front of x^3 is $\frac{1}{2000}$.

(c) Why do you think I gave you this problem in the homework that is mostly on spline interpolation? If you think of the shape of the rod as given by a cubic spline, how would you classify the spline? (Words that are potentially relevant are FREE, CLAMPED, NOT-A-KNOT.)

Problem 2. Consider the function $f(x) = \sqrt{x}$. We want to find a cubic polynomial $S(x)$ that interpolates $f(x)$ on the interval $x \in [1, 4]$ with clamped boundary conditions at both ends.

(a) Clearly, $S(x)$ must have the same values as $f(x)$ at the points $x_0 = 1$ and $x_1 = 4$. What are the clamped boundary conditions for $S'(1)$ and $S'(4)$?

(b) Write the interpolating polynomial $S(x)$ in the form

$$S(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 ,$$

and impose the conditions formulated in part (a) to find the coefficients of $S(x)$.
Hint: I found that $d = \frac{1}{108}$.

(c) The function $S(x)$ that you found in (a) is good not only to approximate the values of the function $f(x)$, but also to approximate the values of the integrals and some derivatives of $f(x)$. Find the numerical value of $\int_1^3 S(x)\,dx$ and compare it with the exact value, $\int_1^3 f(x)\,\mathrm{d}x$; find the absolute and the relative errors.
Hint: You may use that the exact value of the integral is $\int_1^3 \sqrt{x}\,dx = 2\sqrt{3} - \frac{2}{3} \approx 2.79743$.

(d) Find the numerical value of $S'(2)$ and compare it with the exact value, $f'(2)$; find the absolute and the relative errors.

Problem 3. A clamped cubic spline S for a function f is given by

$$S(x) = \begin{cases} S_0(x) = 1 + x + 2x^2 , & \text{for } 0 \le x \le 1 , \\ S_1(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 , & \text{for } 1 \le x \le 2 . \end{cases}$$

The function $f(x)$ is known to satisfy the conditions

$$f'(0) = 1 \qquad \text{and} \qquad f'(2) = 0 .$$

Recall that the first derivatives of the clamped splines at the endpoints (in this case, at $x = 0$ and $x = 2$) are equal to the first derivatives of the original function f at these points.
Find the values of the constants a, b, c, and d in the expression for S.

Problem 4. For each function below, find the limits and determine the corresponding convergence rates as $x \to 0$. Your answers should look like this: $\frac{\cos x - 1}{x^2} = -\frac{1}{2} + O(x^2)$ for $x \to 0$; here O is the "big O" notation introduced in Lecture 5. For hints and examples, see the detailed notes below.

(a) $\dfrac{\mathrm{e}^x - 1}{x}$;

(b) $\dfrac{\sin x}{x}$;

(c) $\dfrac{\mathrm{e}^x - \cos x - x}{x^2}$;

(d) $\dfrac{\cos x - 1 + (x^2/2) - (x^4/24)}{x^6}$.

Hints on rate of convergence of functions. In Problem 4 of this homework you are supposed to find several limits of functions and the rates of convergence. The whole point of finding the order of convergence of a function to its limit is to get a rough idea how the function approaches its limiting value as the argument of the function approaches zero. Here are some examples that will hopefully make things clearer. The main tool in this kind of problems is the Taylor expansion.

Example 1. We know that the Taylor expansion of the function $\cos x$ around 0 is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \ .$$

Clearly, when $x \to 0$, all terms except the first tend to zero, so that

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) = 1$$

As x becomes very close to zero, the term $\frac{x^4}{4!}$ is much smaller than $\frac{x^2}{2!}$ – indeed,

$$\frac{x^4/4!}{x^2/2!} = \text{const} \cdot x^2 \ , \qquad \text{so that} \qquad \lim_{x\to 0} \frac{x^4/4!}{x^2/2!} = \lim_{x\to 0} \left(\text{const} \cdot x^2\right) = 0 \ .$$

This means that, as $x \to 0$, the term $\frac{x^4}{4!}$ becomes negligible in comparison with $\frac{x^2}{2!}$. Note that I did not even compute the constant (just wrote "const"), because the only thing that is important for me here are the powers of x. Similarly, the terms proportional to x^6, x^8, etc., are negligible in comparison with $\frac{x^2}{2!}$. Therefore we obtain

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = 1 - \frac{x^2}{2!} + (\text{terms that go to 0 faster than the term with } x^2) \ ,$$

which allows us to write

$$\cos x = 1 + O(x^2) \ .$$

Note that we do not care about the constant that multiplies x^2 (which in this particular case is equal to $-\frac{1}{2!}$ but, again, that is not important).

Example 2. To find the limit and the rate of convergence of $\frac{\mathrm{e}^x + \cos x - 2 - x}{x^3}$ as $x \to 0$, we use the Taylor expansion of $\cos x$ around 0 (see above) and the Taylor expansion of e^x around 0:

$$\mathrm{e}^x = 1 + \tfrac{x^1}{1!} + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \cdots \ ,$$

so that

$$\begin{aligned}
\frac{\mathrm{e}^x + \cos x - 2 - x}{x^3} &= \tfrac{1}{x^3}\left(1 + \tfrac{x^1}{1!} + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \cdots + 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdots - 2 - x\right) \\
&= \tfrac{1}{x^3}\left(\tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \tfrac{x^5}{5!} + \cdots + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \cdots\right) \\
&= \tfrac{1}{3!} + 2\tfrac{x}{4!} + \tfrac{x^2}{5!} + (\text{terms with even higher powers of } x) \\
&= \tfrac{1}{6} + O(x) \ .
\end{aligned}$$

Using the above calculations, we see right away that $\mathrm{e}^x + \cos x - 2 - x = O(x^3)$ (why?).

Example 3. Using the Taylor expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ ,$$

we obtain

$$\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots \ ,$$

and, therefore,

$$\frac{\ln(1+x^3)}{x^3} = 1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \cdots = 1 + O(x^3) \ .$$

Simple calculations like the ones above is all I expect you to do in Problem 4 in the homework.