MATH 4073 Homework 5 Due 10/8/09

Problem 1. The table below presents the first several terms of a sequence $\{p_n\}_{n=0}^{\infty}$ obtained as a result of an iterative procedure for finding the root of a certain nonlinear equation.

n	p_n
0	1.00000000000000
1	0.87604087217567
2	1.29156974051677
3	1.23258620750928
4	1.19722828956435
5	1.17495576224682
6	1.16012767919729
7	1.14974649721334
8	1.14216234778105
9	1.13642265258200
$\vdots$	$\vdots$

Use the values of p_7, p_8, and p_9 to compute the term $\hat{p}_7$ by using Aitken's Δ^2-method for accelerating the convergence of the sequence $\{p_n\}$. Compare the values of the errors $|p_9 - p|$ and $|\hat{p}_7 - p|$, where p is the exact value of the root of the nonlinear equation which is known to be equal to $p = 1.098612288668110\ldots$.

Problem 2. In this problem you will find approximate solutions of the nonlinear equation $x + \mathrm{e}^{0.00001x} = 100$.

(a) You can give a *very rough estimate* of the solution thinking like this. Clearly, the left-hand side of the equation is an increasing function of x. If $x = 100$, then the left-hand side is $100 + \mathrm{e}^{0.00001 \cdot 100} = 100 + \mathrm{e}^{0.01}$, which is a little more than 101, so that the root we are looking for must be a little less than 100. Since x in $\mathrm{e}^{0.00001x}$ is multiplied by the very small number 0.00001, for $x \approx 100$, we will have $\mathrm{e}^{0.00001x} \approx 1$. Continue...

(b) Now follow the ideas of part (a) and use Taylor expansion of the exponent to obtain an approximation of the solution that is better than the one obtained in part (a).

(c) Run the code `newton.m` (or your code `newton_m.m` from Homework 4) to find the exact root of the nonlinear equation; attach your Matlab printout.

(d) Find the absolute and the relative errors of the approximate solutions found in parts (a) and (b).

Problem 3. Consider the polynomial equation

$$-x^4 + \frac{13}{6}x^2 - \frac{1}{2} = 0 \ .$$

(a) Find the exact roots of this equation by hand.

Hint: Note that the equation has only even powers of x.

(b) Perform by hand two steps of a Newton iteration for this equation starting from $p_0 = 1$. What do you notice? (The answer to this question must be totally obvious; if it is not, check your calculations.) What will Newton iteration produce in this case if we assume that the computer has infinite accuracy (i.e., there is no numerical error).

(c) Now run the code `newton.m` verbosely for this equation, starting from $p_0 = 1.0$; allow enough maximum number of iterations; choosing tolerance 10^{-14} is safe (in Matlab you get 10^{-14} by typing `1e-14`). What do you observe? Compare your theoretical prediction from part (b) with your observations from running the Matlab code. Explain briefly what happened.

(d) Now run the code `newton.m` for this equation, starting from the following values of p_0: 0.93, 0.94, 0.96, 0.97, 0.999, 0.9999, 0.99999, 1.00001, 1.0001, 1.001; make a table giving the values of the initial points and the value of the root found in each case. What is the moral learned from this part of the problem?

Problem 4. In this problem you will study in detail the piecewise-linear interpolation of the function $f(x) = \frac{1}{x}$. on the interval $[1, 2]$, and then on the interval $[1, 3]$. The graphs of the function and the Lagrange interpolating polynomial on the interval $[1, 2]$ are shown in Figure 1.

(a) Find the first order Lagrange polynomial $P_1(x)$ of $f(x) = \frac{1}{x}$ that passes through the points $(1, f(1))$ and $(2, f(2))$.

(b) Let

$$E := \max_{x \in [1,2]} |f(x) - P_1(x)|$$

be the *true* error of the first order Lagrange interpolation. Find the numerical value of E.

Hint: You first have to find the value x^* of the argument that maximizes the expression $|f(x) - P_1(x)|$. Note that f is concave up, so that the graph of P_1 lies above the graph of f, therefore $|f(x) - P_1(x)| = P_1(x) - f(x)$.

(c) Find the error bound of the linear interpolation on $[1, 2]$ given by Theorem 3.3 in Section 3.1 of the book. Note that you do not know the value of $\xi(x)$ in this bound, so you will have to take the maximum of the absolute value of the derivative in this

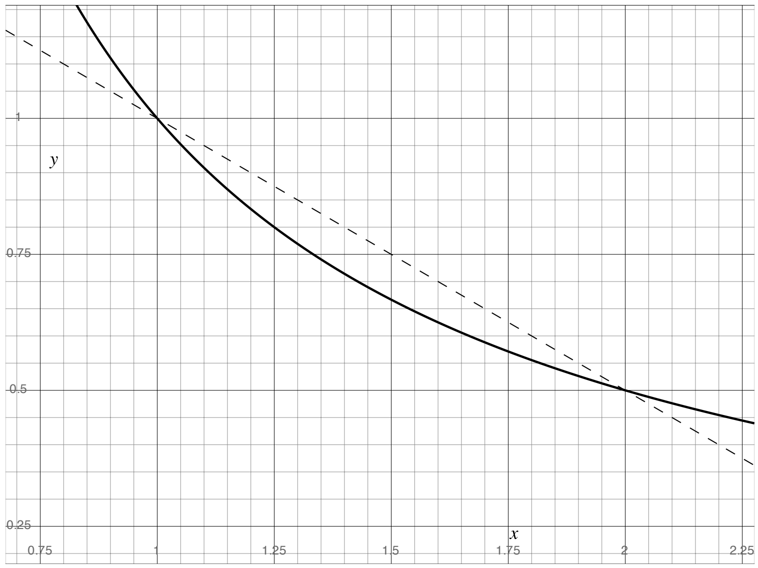

Figure 1: Linear interpolation of $f(x) = \frac{1}{x}$ on the interval $[1, 2]$ (the function f is plotted with a thick solid line, and the interpolating polynomial with a dashed line).

bound over the whole interval $[1, 2]$. Separately, you will have to find the maximum value of the absolute value of the product of $(x - x_j)$ terms (look at the sign of each $(x - x_j)$ and get rid of the absolute values before taking derivatives). Find the exact value of this bound, and then compute its numerical value. Compare with the exact value of the error found in part (b).

(d) Now find the Lagrange interpolating polynomial of f over the interval $[2, 3]$, and write your results from parts (a) and (c) together in the form

$$P_{\text{piecewise-linear}}(x) = \begin{cases} b_1 x + c_1 \ , & x \in [1, 2] \ , \\ b_2 x + c_2 \ , & x \in [2, 3] \ . \end{cases}$$

(e) Use your result from part (d) to compute $P_{\text{piecewise-linear}}(1.25)$, and compare its value with $f(1.25)$.

(f) Finally, compute the Taylor series of f around $x_0 = 1$. Does it converge for $x = 2$?

Problem 5. Use Newton's divided differences to find the interpolating cubic polynomial for the function $f(x) = \frac{1}{x}$ if $x_0 = \frac{1}{2}$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. Write the divided differences in a table; some of the values of the divided differences are computed below – you can use them to check your computations.

x_i	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order
$x_0 = \frac{1}{2}$	$f[x_0] = ?$			
		$f[x_0, x_1] = -2$		
$x_1 = 1$	$f[x_1] = 1$		$f[x_0, x_1, x_2] = ?$	
		$f[x_1, x_2] = -\frac{1}{2}$		$f[x_0, x_1, x_2, x_3] = ?$
$x_2 = 2$	$f[x_2] = ?$		$f[x_1, x_2, x_3] = \frac{1}{6}$	
		$f[x_2, x_3] = ?$		
$x_3 = 3$	$f[x_3] = ?$			

Compute the value of the interpolating polynomial at $x = 2.5$, and compare it with the exact value $f(2.5)$.

Remark: The figure below shows the function (with a thick solid line) and the interpolating cubic polynomial (with a dashed line). Note the values of these two functions at the points x_j.