MATH 3423 Homework 5 Due Fri, 10/4/13

Useful facts about the hyperbolic sine and cosine functions:

$$\sinh z := \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2}\,, \quad \cosh z := \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2}\,, \quad \frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \cosh z\,, \quad \frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z\,.$$

Problem 1. Consider the function $f(z) = \dfrac{1}{z^3(z-5)}$.

(a) Find and classify the singularities of $f(z)$.

(b) Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $0 < |z| < 5$.

(c) Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $5 < |z|$.

(d) Compute the residue of $f(z)$ at $a = 0$. If you use some of the previously obtained results to answer this question, please specify clearly what you used.

(e) Compute the residue of $f(z)$ at $a = 5$. (I am asking you only about the value of the residue, there is no need to find the Laurent series.)

Problem 2. Consider the function $f(z) = \dfrac{\mathrm{e}^{3z}}{(z-5)^4}$.

(a) What is the nature of the singularity of $f(z)$ at $a = 5$?

Hint: You may use the Lemma on page 17.5 of the lecture notes to answer this question.

(b) Find the Laurent expansion of $f(z)$ about $a = 5$.

Hint: Look at Example 5 on page 908 of the book.

(c) Find the residue of $f(z)$ at $a = 5$ from your result in part (c).

(d) Compute the residue of $f(z)$ at $a = 5$ by using the formula on page 17.7 of the lecture notes.

Problem 3. The function $f(z) = \dfrac{1-\cos z}{\sin^3 z}$ has poles at $z = k\pi$ for each $k \in \mathbb{Z}$.

(a) The pole $a = 0$ is simple (i.e., of order 1), which can easily be seen by expanding $\cos z$ and $\sin z$ in Taylor series:

$$f(z) = \frac{1 - (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots)}{(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots)^3} = \frac{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots}{z^3(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots)^3} = \frac{\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots}{z(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots)^3}\,,$$

therefore $f(z) = \dfrac{g(z)}{z}$, where $g(z)$ is analytic at 0 (because it is equal to the ratio of the analytic functions $h(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots$ and $w(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$, with $w(0) \neq 0$). Use the formula for the residues (see page 17.7 of the lecture notes) to find the value of the residue of $f(z)$ at $a = 0$.

(b) The pole $a = \pi$ is of order 3. To see this, one needs to expand $f(z)$ in powers of $(z - \pi)$. Since you probably don't know the expansions of $\cos z$ and $\sin z$ about 0, you can instead introduce a new variable, $\zeta = z - \pi$, and use the standard trigonometric facts $\cos(\pi + \zeta) = -\cos\zeta$, and $\sin(\pi + \zeta) = -\sin\zeta$ in order to write

$$\frac{1-\cos z}{\sin^3 z} = \frac{1-\cos(\pi+\zeta)}{\sin^3(\pi+\zeta)} = -\frac{1+\cos\zeta}{\sin^3\zeta} = -\frac{1+\cos\zeta}{\left(\zeta - \frac{\zeta^3}{3!} + \frac{\zeta^5}{5!} - \cdots\right)^3}$$

$$= -\frac{1+\cos\zeta}{\zeta^3\left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \cdots\right)^3} =: \frac{g(\zeta)}{\zeta^3} \ .$$

Since the function $g(\zeta) = -\frac{1+\cos\zeta}{\left(1-\frac{\zeta^2}{3!}+\frac{\zeta^4}{5!}-\cdots\right)^3}$ is analytic in a neighborhood of 0, we conclude that $\frac{g(\zeta)}{\zeta^3}$ has a pole of order 3 at 0, which is equivalent to saying that $\frac{1-\cos z}{\sin^3 z}$ has a pole of order 3 at $\pi + 0 = \pi$ (because $z = \pi + \zeta$).

Now the general theory tells us that the residue of $\frac{1-\cos z}{\sin^3 z}$ at $a = \pi$ can be computed by using the formula

$$\operatorname{Res}\left(\frac{1-\cos z}{\sin^3 z}, \pi\right) = \frac{1}{2!}\lim_{z\to\pi}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(z-\pi)^3\,\frac{1-\cos z}{\sin^3 z}\right] \ .$$

Using this formula go compute the residue by hand, however, is very difficult and time-consuming. Instead of doing it this way, one can use the Taylor expansions of sine and cosine about 0 like what we did above. Using again $\zeta = z - \pi$, we have

$$\frac{1-\cos z}{\sin^3 z} = -\frac{1+\cos\zeta}{\sin^3\zeta} = -\frac{1+\left(1-\frac{\zeta^2}{2!}+\frac{\zeta^4}{4!}-\cdots\right)}{\left(\zeta-\frac{\zeta^3}{3!}+\frac{\zeta^5}{5!}-\cdots\right)^3} = -\frac{2-\frac{\zeta^2}{2!}+\frac{\zeta^4}{4!}-\cdots}{\zeta^3\left(1-\frac{\zeta^2}{3!}+\frac{\zeta^4}{5!}-\cdots\right)^3} \ .$$

We want to find the residue, i.e., the coefficient in front of ζ^{-1} (or, equivalently, in front of $(z-\pi)^{-1}$). To this end, we have to find the expansion of $-\dfrac{2-\frac{\zeta^2}{2!}+\frac{\zeta^4}{4!}-\cdots}{\left(1-\frac{\zeta^2}{3!}+\frac{\zeta^4}{5!}-\cdots\right)^3}$ up to order ζ^2, because the ζ^2 term divided by the ζ^3 will produce the desired residue. Since we do not need the higher-order terms, we can write

$$\left(1-\frac{\zeta^2}{3!}+\frac{\zeta^4}{5!}-\cdots\right)^3 = \left[1-\zeta^2\left(\frac{1}{3!}+\frac{\zeta^2}{5!}-\cdots\right)\right]^3 = \left[1-\zeta^2\left(\frac{1}{3!}+\frac{\zeta^2}{5!}-\cdots\right)\right]^3$$

$$= 1 - 3\,\zeta^2\left(\frac{1}{3!}+\frac{\zeta^2}{5!}-\cdots\right) + 3\,\zeta^4\left(\frac{1}{3!}+\cdots\right)^2 - \zeta^6\left(\frac{1}{3!}+\cdots\right)^3 \approx 1 - \frac{\zeta^2}{2} \ ,$$

and then use the geometric series formula:

$$\frac{1}{\left(1-\frac{\zeta^2}{3!}+\frac{\zeta^4}{5!}-\cdots\right)^3} \approx \frac{1}{1-\frac{\zeta^2}{2}} = 1+\frac{\zeta^2}{2}+\left(\frac{\zeta^2}{2}\right)^2+\cdots \approx 1+\frac{\zeta^2}{2}\ .$$

Now we put all this together:

$$-\frac{2-\frac{\zeta^2}{2!}+\frac{\zeta^4}{4!}-\cdots}{\left(1-\frac{\zeta^2}{3!}+\frac{\zeta^4}{5!}-\cdots\right)^3} \approx -\left(2-\frac{\zeta^2}{2!}\right)\left(1+\frac{\zeta^2}{2}\right) = \cdots\ .$$

Continue this calculation in order to obtain the desired residue.

(c) **[Food for Thought only – not to be turned in!]**
The function $f(z)$ satisfies $f(z+2\pi k) = f(z)$ for any $k \in \mathbb{Z}$. Use this and your result from part (a) to find the residues at $a = 2k\pi$, $k \in \mathbb{Z}$.

(d) **[Food for Thought only – not to be turned in!]**
Directly from the definition, prove that $f(-z) = -f(z)$. What does this imply about the values of the residues at $a = (2k+1)\pi$, $k \in \mathbb{Z}$?
Hint: Use your results from parts (b) and (c).

Problem 4. Consider the function $f(z) = z^3 \cosh \frac{1}{z}$.

(a) Derive the Taylor series of $\cosh z$.

(b) Find the Laurent expansion of $f(z)$ about the point $a = 0$.

(c) The point $a = 0$ is a singularity of $f(z)$. What kind of singularity?

(d) Find $\oint_C f(z)\,\mathrm{d}z$, where C is the ellipse $\frac{x^2}{3^2}+\frac{y^2}{2^2} = 1$, traversed in positive direction.

Problem 5.

(a) Use the formula for the sum of a geometric series to find the sum of the series $\sum_{n=0}^{\infty} r^n \mathrm{e}^{in\theta}$ (where $r > 0$ and θ are real numbers). For what values of r and θ is your result valid?

(b) Use the formula derived in part (a) to prove that

$$\sum_{n=0}^{\infty} r^n \cos(n\theta) = \frac{1-r\cos\theta}{1+r^2-2r\cos\theta}\ .$$

For what values of r and θ is this formula valid?

(c) Use the formula for the sum of a geometric series to obtain a simple expression for the sum of the series $\sum_{n=1}^{\infty} n r^{n-1} \mathrm{e}^{\mathrm{i}(n-1)\theta}$.

Hint: How can you get nz^{n-1} from z^n?

Problem 6. Find all Laurent expansions of $f(z) = \dfrac{1}{z-2}$ about the point $a = i$.