

USEFUL FACTS ABOUT THE HYPERBOLIC SINE AND COSINE FUNCTIONS:

$$\sinh z := \frac{e^z - e^{-z}}{2}, \quad \cosh z := \frac{e^z + e^{-z}}{2}, \quad \frac{d}{dz} \sinh z = \cosh z, \quad \frac{d}{dz} \cosh z = \sinh z.$$

Problem 1. Consider the function $f(z) = \frac{1}{z^3(z-5)}$.

- Find and classify the singularities of $f(z)$.
- Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $0 < |z| < 5$.
- Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $5 < |z|$.
- Compute the residue of $f(z)$ at $a = 0$. If you use some of the previously obtained results to answer this question, please specify clearly what you used.
- Compute the residue of $f(z)$ at $a = 5$. (I am asking you only about the value of the residue, there is no need to find the Laurent series.)

Problem 2. Consider the function $f(z) = \frac{e^{3z}}{(z-5)^4}$.

- What is the nature of the singularity of $f(z)$ at $a = 5$?
Hint: You may use the Lemma on page 17.5 of the lecture notes to answer this question.
- Find the Laurent expansion of $f(z)$ about $a = 5$.
Hint: Look at Example 5 on page 908 of the book.
- Find the residue of $f(z)$ at $a = 5$ from your result in part (c).
- Compute the residue of $f(z)$ at $a = 5$ by using the formula on page 17.7 of the lecture notes.

Problem 3. The function $f(z) = \frac{1 - \cos z}{\sin^3 z}$ has poles at $z = k\pi$ for each $k \in \mathbb{Z}$.

- The pole $a = 0$ is simple (i.e., of order 1), which can easily be seen by expanding $\cos z$ and $\sin z$ in Taylor series:

$$f(z) = \frac{1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right)}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)^3} = \frac{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \dots}{z^3 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)^3} = \frac{\frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots}{z \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)^3},$$

therefore $f(z) = \frac{g(z)}{z}$, where $g(z)$ is analytic at 0 (because it is equal to the ratio of the analytic functions $h(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots$ and $w(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$, with $w(0) \neq 0$). Use the formula for the residues (see page 17.7 of the lecture notes) to find the value of the residue of $f(z)$ at $a = 0$.

- (b) The pole $a = \pi$ is of order 3. To see this, one needs to expand $f(z)$ in powers of $(z - \pi)$. Since you probably don't know the expansions of $\cos z$ and $\sin z$ about 0, you can instead introduce a new variable, $\zeta = z - \pi$, and use the standard trigonometric facts $\cos(\pi + \zeta) = -\cos \zeta$, and $\sin(\pi + \zeta) = -\sin \zeta$ in order to write

$$\begin{aligned} \frac{1 - \cos z}{\sin^3 z} &= \frac{1 - \cos(\pi + \zeta)}{\sin^3(\pi + \zeta)} = -\frac{1 + \cos \zeta}{\sin^3 \zeta} = -\frac{1 + \cos \zeta}{\left(\zeta - \frac{\zeta^3}{3!} + \frac{\zeta^5}{5!} - \dots\right)^3} \\ &= -\frac{1 + \cos \zeta}{\zeta^3 \left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \dots\right)^3} =: \frac{g(\zeta)}{\zeta^3}. \end{aligned}$$

Since the function $g(\zeta) = -\frac{1 + \cos \zeta}{\left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \dots\right)^3}$ is analytic in a neighborhood of 0, we conclude that $\frac{g(\zeta)}{\zeta^3}$ has a pole of order 3 at 0, which is equivalent to saying that $\frac{1 - \cos z}{\sin^3 z}$ has a pole of order 3 at $\pi + 0 = \pi$ (because $z = \pi + \zeta$).

Now the general theory tells us that the residue of $\frac{1 - \cos z}{\sin^3 z}$ at $a = \pi$ can be computed by using the formula

$$\text{Res} \left(\frac{1 - \cos z}{\sin^3 z}, \pi \right) = \frac{1}{2!} \lim_{z \rightarrow \pi} \frac{d^2}{dz^2} \left[(z - \pi)^3 \frac{1 - \cos z}{\sin^3 z} \right].$$

Using this formula to compute the residue by hand, however, is very difficult and time-consuming. Instead of doing it this way, one can use the Taylor expansions of sine and cosine about 0 like what we did above. Using again $\zeta = z - \pi$, we have

$$\frac{1 - \cos z}{\sin^3 z} = -\frac{1 + \cos \zeta}{\sin^3 \zeta} = -\frac{1 + \left(1 - \frac{\zeta^2}{2!} + \frac{\zeta^4}{4!} - \dots\right)}{\left(\zeta - \frac{\zeta^3}{3!} + \frac{\zeta^5}{5!} - \dots\right)^3} = -\frac{2 - \frac{\zeta^2}{2!} + \frac{\zeta^4}{4!} - \dots}{\zeta^3 \left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \dots\right)^3}.$$

We want to find the residue, i.e., the coefficient in front of ζ^{-1} (or, equivalently, in front of $(z - \pi)^{-1}$). To this end, we have to find the expansion of $-\frac{2 - \frac{\zeta^2}{2!} + \frac{\zeta^4}{4!} - \dots}{\left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \dots\right)^3}$

up to order ζ^2 , because the ζ^2 term divided by the ζ^3 will produce the desired residue. Since we do not need the higher-order terms, we can write

$$\begin{aligned} \left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \dots\right)^3 &= \left[1 - \zeta^2 \left(\frac{1}{3!} + \frac{\zeta^2}{5!} - \dots\right)\right]^3 = \left[1 - \zeta^2 \left(\frac{1}{3!} + \frac{\zeta^2}{5!} - \dots\right)\right]^3 \\ &= 1 - 3\zeta^2 \left(\frac{1}{3!} + \frac{\zeta^2}{5!} - \dots\right) + 3\zeta^4 \left(\frac{1}{3!} + \dots\right)^2 - \zeta^6 \left(\frac{1}{3!} + \dots\right)^3 \approx 1 - \frac{\zeta^2}{2}, \end{aligned}$$

and then use the geometric series formula:

$$\frac{1}{\left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \dots\right)^3} \approx \frac{1}{1 - \frac{\zeta^2}{2}} = 1 + \frac{\zeta^2}{2} + \left(\frac{\zeta^2}{2}\right)^2 + \dots \approx 1 + \frac{\zeta^2}{2}.$$

Now we put all this together:

$$-\frac{2 - \frac{\zeta^2}{2!} + \frac{\zeta^4}{4!} - \dots}{\left(1 - \frac{\zeta^2}{3!} + \frac{\zeta^4}{5!} - \dots\right)^3} \approx -\left(2 - \frac{\zeta^2}{2!}\right)\left(1 + \frac{\zeta^2}{2}\right) = \dots.$$

Continue this calculation in order to obtain the desired residue.

(c) **[Food for Thought only – not to be turned in!]**

The function $f(z)$ satisfies $f(z + 2\pi k) = f(z)$ for any $k \in \mathbb{Z}$. Use this and your result from part (a) to find the residues at $a = 2k\pi$, $k \in \mathbb{Z}$.

(d) **[Food for Thought only – not to be turned in!]**

Directly from the definition, prove that $f(-z) = -f(z)$. What does this imply about the values of the residues at $a = (2k + 1)\pi$, $k \in \mathbb{Z}$?

Hint: Use your results from parts (b) and (c).

Problem 4. Consider the function $f(z) = z^3 \cosh \frac{1}{z}$.

(a) Derive the Taylor series of $\cosh z$.

(b) Find the Laurent expansion of $f(z)$ about the point $a = 0$.

(c) The point $a = 0$ is a singularity of $f(z)$. What kind of singularity?

(d) Find $\oint_C f(z) dz$, where C is the ellipse $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, traversed in positive direction.

Problem 5.

(a) Use the formula for the sum of a geometric series to find the sum of the series $\sum_{n=0}^{\infty} r^n e^{in\theta}$ (where $r > 0$ and θ are real numbers). For what values of r and θ is your result valid?

(b) Use the formula derived in part (a) to prove that

$$\sum_{n=0}^{\infty} r^n \cos(n\theta) = \frac{1 - r \cos \theta}{1 + r^2 - 2r \cos \theta}.$$

For what values of r and θ is this formula valid?

- (c) Use the formula for the sum of a geometric series to obtain a simple expression for the sum of the series $\sum_{n=1}^{\infty} nr^{n-1}e^{i(n-1)\theta}$.

Hint: How can you get nz^{n-1} from z^n ?

Problem 6. Find all Laurent expansions of $f(z) = \frac{1}{z-2}$ about the point $a = i$.