MATH 5763 **Homework 4** **Due Thu, 2/14/2019**

Problem 1. This problem is a continuation of Problems 4 and 5 of Homework 3. In those problems you considered a Markov chain with five states, and after relabeling the states the transition matrix of the Markov chain became

$$\mathbf{P} = \left(\begin{array}{c|c|c} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array}\right) = \left(\begin{array}{cc|c|cc} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{array}\right);$$

here $\mathbf{0}$ denotes a matrix of appropriate size with zero entries, while a star denotes an arbitrary matrix of appropriate size. You proved that $\mathbf{C}_1$ and $\mathbf{C}_2$ are stochastic matrices, while $\mathbf{T}$ is not.

The form in which we wrote the transition probabilities $\mathbf{P}$ (after relabeling the states) is very convenient for studying the long-term behavior of the Markov chain because of the following fact:

$$\mathbf{P}^n = \left(\begin{array}{c|c|c} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array}\right)^n = \left(\begin{array}{c|c|c} \mathbf{C}_1^n & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2^n & \mathbf{0} \\ \hline * & * & \mathbf{T}^n \end{array}\right).$$

The stars are generally non-zero entries that reflect the probabilities with which the Markov chain "leaks out" of the transient states 4 and 5 and goes to one of the closed sets $\{1,2\}$ or $\{3\}$ of recurrent states.

In this problem you will compute the stationary distribution(s) of this Markov chain.

(a) Write down the six equations that the stationary distribution $\boldsymbol{\pi} = (\pi_1\ \pi_2\ \pi_3\ \pi_4\ \pi_5)$ satisfies. You will very easily see from the linear system that $\pi_4 = 0$ and $\pi_5 = 0$ (but you have to obtain this from the system!). How do you explain this fact without doing any calculations?

(b) Find the most general form of a stationary distribution $\boldsymbol{\pi}$ from your solution in part (a). Your solution for $\boldsymbol{\pi}$ will be non-unique; namely, you will find that $\boldsymbol{\pi}$ will depend on one parameter. Discuss this observation in the light of the Ergodic Theorem, and explain it physically.

(c) Can you suggest a method for computing all stationary distributions of the Markov chain in this problem without ever solving a system of six equations simultaneously? Explain briefly how you are going to do it, and why your method will work.

Problem 2. This problem is a continuation of Problem 1.

(a) Consider only the irreducible matrix $\mathbf{C}_1$ containing only the recurrent states 1 and 2. Let μ_i be the average number of transitions needed by the process, starting from state i, to return to i for the first time (sometimes μ_i is called the *mean recurrence time* of state i). In part (a) of Problem 5 of Homework 3 you found the probabilities $\rho_{ii}^{(n)}$ of returning to state i from the initial state i for the first time in exactly n steps (for $i \in \{1, 2\}$); these values are also given on page 82 of Lefebvre's book. Use your results to compute the expected values μ_1 and μ_2 of the first return times. Are the states 1 and 2 positive recurrent or null recurrent? Is your result consistent with the general theory? Explain briefly.

Hint: The following trick is very useful for evaluating sums: differentiating with respect to q both sides of the formula for the sum of a geometric series, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ (valid for $|q| < 1$), one obtains $\sum_{n=1}^{\infty} n q^{n-1} = \frac{1}{(1-q)^2}$ (in the sum in the left-hand side one can start the summation from 0, but the term with $n = 0$ is equal to zero). Incidentally, differentiating one more time, one can obtain an expression for $\sum_{n=2}^{\infty} n(n-1) q^n$, from which $\sum_{n=2}^{\infty} n^2 q^n$ can be found, etc.

(b) Consider only the matrix $\mathbf{C}_1$ containing the recurrent states 1 and 2. Since these two states form an irreducible set, the Ergodic Theorem guarantees that it has a unique stationary distribution $\widetilde{\boldsymbol{\pi}} = (\widetilde{\pi}_1 \ \widetilde{\pi}_2)$. Find $\widetilde{\boldsymbol{\pi}}$.

(c) Your results in (a) and (b) are related. How? Discuss this briefly in the light of the general theory.

(d) In the rest of this problem you will consider the transient states 4 and 5. Recall that the indicator function of an event $A \subseteq \Omega$ is a random variable $I_A : \Omega \to \{0, 1\}$ defined as

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \ , \\ 0 & \text{if } \omega \notin A \ . \end{cases}$$

Let j be a transient state,

$$Y_j := \sum_{n=0}^{\infty} I_{\{X_n = j\}}$$

be the number of times the Markov chain visits it, and

$$E[Y_j \,|\, X_0 = k] = E\left[\sum_{n=0}^{\infty} I_{\{X_n = j\}} \,\middle|\, X_0 = k \right]$$

be the expected number of times the chain visits the transient state j if initially it is in the transient state k. Show that

$$E[Y_j \,|\, X_0 = k] = \sum_{n=0}^{\infty} p_{kj}^{(n)} = \left(\sum_{n=0}^{\infty} \mathbf{P}^{(n)} \right)_{kj} = \left(\sum_{n=0}^{\infty} \mathbf{P}^{n} \right)_{kj} = \left((\mathbf{I} - \mathbf{T})^{-1} \right)_{kj} \ ,$$

where $\mathbf{I}$ is the unit matrix of appropriate size.

Hint: Use the fact that, for a matrix $\mathbf{A}$, if the "geometric series" $\sum_{n=0}^{\infty} \mathbf{A}^n$ converges, its sum is equal to $\sum_{n=0}^{\infty} \mathbf{A}^n = (\mathbf{I} - \mathbf{A})^{-1}$. Note that for numbers (i.e., 1×1 matrices) this becomes the well-known formula.

Remark: If $\mathbf{T}$ corresponds to the transient states only, there is a theorem that guarantees that the matrix $(\mathbf{I} - \mathbf{T})$ is invertible.

(e) I did the math, and obtained

$$\mathbf{T} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad \mathbf{I} - \mathbf{T} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \qquad (\mathbf{I} - \mathbf{T})^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & \frac{3}{2} \end{pmatrix}.$$

Discuss the meaning of each entry of $(\mathbf{I} - \mathbf{T})^{-1}$ in the light of what you proved in part (d).

Remark: In the definition of Y_j, we included $n = 0$ as a possibility – this means that, if we start at state j, we add 1 to the sum defining Y_j – this is the term with $n = 0$, i.e., $I_{\{X_0]j\}}$ (given that $X_0 = j$). Therefore, having defined Y_j this way, we must always have $[(\mathbf{I} - \mathbf{T})^{-1}]_{jj} \geq 1$, as in this particular example.

Problem 3. In this and in the next problem you will consider a particular case of the gambler's ruin problem. It will be useful if you review the material from pages 85–88 and 100–104 of the book. In particular, you need to know the definitions of:

- the probability $\rho_{ij}^{(n)}$ of first visit of state j starting from state i in exactly n steps, where $n \geq 1$ (page 81);
- the probability $f_{ij} = \sum_{n=1}^{\infty} \rho_{ij}^{(n)}$ of eventually visiting state j starting from state i (page 87);
- the probability $r_i(C)$ of eventually entering the closed irreducible set C of recurrent states if initially the system is in the transient state i (page 100).

Consider a four-state gambler's ruin chain with state space $\mathscr{X} = \{0, 1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw a picture with the states and arrows corresponding to non-zero transition probabilities p_{ij} between them. Identify the transient and the recurrent states, as well as the closed and irreducible classes of recurrent states.

(b) Compute the probability $\rho_{10}^{(n)}$ for all $n \geq 1$.

Hint: This is easy for the particular transition matrix in this problem! For example, it is clear that $\rho_{10}^{(n)} = 0$ for all even values of n.

(c) Use your result from (b) to show that $f_{10} = \frac{4}{13}$.

(d) Find the probability f_{13}. Explain briefly your reasoning.

Hint: If you think about the meaning of this probability, you won't need to do any calculations!

(e) Show that $f_{11} = \rho_{11}^{(2)}$ and use this to compute f_{11}.

(f) Determine f_{22}.

(g) Given that $X_0 = 1$, what is the expected number of time that the chain will return to state 1 in the subsequent steps by using the fact that, if the chain is at state i at time $n = 0$, the expected number of subsequent visits to a transient state j is equal to $\dfrac{f_{ij}}{1 - f_{jj}}$. (You can take this fact for granted, but think about its meaning in the light of Equation (3.41) on page 88 of Lefevbre's book.)

(h) Answer the same question as in (g) but this time using the method you derived in Problem 2(d,e) above.

Problem 4. In this problem you will find the probability of ruin of a gambler for the same transition probability matrix as in Problem 3, but using different methods.

Recall Theorem 3.2.2 of the book according to which, if C is a closed and irreducible set of recurrent states, and D is the set of all transient states, and $i \in D$ is a transient state, then the probability $r_i(C)$ of the chain to enter the set C eventually if $X_0 = i$ is the smallest nonnegative solution of the system

$$r_i(C) = \sum_{j \in D} p_{ij}\, r_j(C) + \sum_{j \in C} p_{ij} \ , \qquad \text{for all } \ i \in D \ .$$

Moreover, if D is a finite set (as in this problem), then the solution is unique.

(a) Write down this system of equations for the probabilities $r_i(\{0\})$ of eventual ruin starting from state i.

(b) Solve the system derived in (a).

(c) Compare your results with your results from Problem 3. Please be specific what numbers you are comparing.

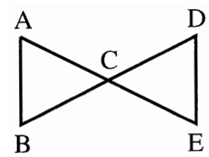
Problem 5. A *(finite, simple, undirected) graph* G is a finite collection of vertices V and a collection of edges E where each edge connects two different vertices, and any two vertices are connected by at most one edge. We write $v_1 \sim v_2$ if the vertices v_1 and v_2 are *adjacent*, i.e., if there is an edge connecting v_1 and v_2.

Consider the Markov chain whose states are the vertices of the graph. At each time interval, the chain chooses a new state randomly from among the states adjacent to the current state, with equal probability. In other words, if at time n the chain is in state i, at time $n+1$ it can be at any of the states adjacent to i, with equal probabilities. (Note that the Markov chain is not allowed to be in the same state in two consecutive times, i.e., the diagonal elements of the transition matrix are all zero.) This Markov chain is called *simple random walk on the graph.* Let $|E|$ be the total number of edges.

(a) Show that the stationary distribution of a simple Markov chain on a graph is given by $\pi_i = \dfrac{d_i}{2|E|}$, where d_i is the *degree* of the vertex i, i.e., the number of edges incident to the vertex i. In the figure in part (b) below, $d_{\mathrm{A}} = d_{\mathrm{B}} = d_{\mathrm{D}} = d_{\mathrm{E}} = 2$, and $d_{\mathrm{C}} = 4$.

Hint: Just check that $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ and the normalization condition.

(b) Write the transition probability matrix $\mathbf{P}$ for the "bowtie" graph in the picture below.



(c) Find the stationary distribution $\boldsymbol{\pi}$ of the simple random walk on the graph from (b) by using the result from (a).

Remark: To make sure that you understood all definitions, check by hand that $\boldsymbol{\pi}$ is a left eigenvector of $\mathbf{P}$ with eigenvalue 1.

(d) In the long run, what fraction of time is the system going to spend in state A?

(e) Find the expected time of first return to each of the five states of the "bowtie" graph.

Hint: The general theory makes this part of the problem extremely easy.

Food for Thought Problem 1. A *linear homogeneous recurrence relation of order d with constant coefficients* is an equation of the form

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + \cdots + b_d x_{n-d} \ , \tag{1}$$

where the d coefficients b_1, b_2, …, b_d are constants. Solving such relations is very similar to solving linear homogeneous ordinary differential equations of order d with constant coefficients. Similarly to the case of differential equations, one talks about a *general solution* of (1) (the general solution contains d arbitrary constants), and for the *particular solution* of (1), which satisfies not only (1), but also *initial conditions*,

$$x_0 = a_0 \ , \ x_1 = a_1 \ , \ \ldots \ , \ x_{d-1} = a_{d-1} \ . \tag{2}$$

To find the general solution of (1), set $x_j = \lambda^j$ in the equation to obtain (after dividing by λ^{n-d}) the equation

$$P(\lambda) = \lambda^d - b_1\lambda^{d-1} - b_2\lambda^{d-2} - \cdots - b_d = 0 \ ,$$

called the *characteristic equation* of (1) (the polynomial P is called the *characteristic polynomial* of (1)). Find all roots λ_j of the characteristic equation.

If all the roots of the are real and distinct (i.e., the roots are $\lambda_1, \ldots \lambda_d$ with $\lambda_j \in \mathbb{R}$ and $\lambda_i \neq \lambda_j$ for $i \neq j$), then the general solution of (1) has the form

$$x_n = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_d\lambda_d^n \ ,$$

where C_1, C_2, …, C_d are arbitrary constants. To find the particular solutions that satisfies both the relation (1) and the initial condition (2), one needs to express the constants C_1, C_2, …, C_d through the initial conditions a_0, a_1, …, a_{d-1}.

If $\lambda_j \in \mathbb{R}$ is a real root of the characteristic equation with multiplicity s_j (i.e., if the characteristic polynomial contains the factor $(\lambda - \lambda_j)^{s_j}$), then the corresponding term in the general solution x_n of the recurrence relation (1) is

$$\left(C_1^{(j)} + C_2^{(j)} n + C_3^{(j)} n^2 + \cdots C_{s_j}^{(j)} n^{s_j - 1} \right) \lambda_j^n \ ,$$

where $C_k^{(j)}$ are arbitrary constants. For example, if $P(\lambda) = (\lambda - 5)\lambda^2(\lambda + 7)^3$, then the general solution of the corresponding recurrence relation is

$$x_n = C_1 5^n + C_2 + C_3 n + (C_4 + C_5 n + C_6 n^2)(-7)^n$$

(after relabeling the arbitrary constants).

(a) The Fibonacci numbers F_n are defined by

$$F_0 = 1 \ , \quad F_1 = 1 \ , \qquad F_n = F_{n-1} + F_{n-2} \ \text{ for } n \geq 2 \ .$$

First few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …. Find an explicit formula for F_n.

(b) Let $p \in (0, 1)$ be a constant and $f : \mathbb{Z} \to \mathbb{R}$ be a function. Find all functions that satisfy the relation

$$f(n) = (1 - p) f(n - 1) + p f(n + 1) \ .$$

You will have to consider the cases $p \neq \frac{1}{2}$ and $p = \frac{1}{2}$ separately.