MATH 5763 Homework 4 Due Fri, 2/25/11

Problem 1. In Problem 2 of Homework 3 you constructed the transition probability matrix $\mathbf{P}$ related to the number of newspapers in the newspaper pile at Brandon's house. The space state of this Markov chain consists of five states, $\{0, 1, 2, 3, 4\}$, and the transition probability matrix is

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Is the Markov chain described by $\mathbf{P}$ ergodic? Explain briefly your reasoning.

(b) Is the Markov chain described by $\mathbf{P}$ irreducible? Explain briefly.

(c) Find the stationary distribution $\boldsymbol{\pi}$ of the Markov chain. Please write your calculations legibly.

(d) If Y is a discrete random variable taking values in the set $\{0, 1, 2, 3, 4\}$ with probabilities $\mathbb{P}(Y = i) = \pi_i$ (where π_i is the ith component of the stationary distribution $\boldsymbol{\pi}$ found in part (c)), find $\mathbb{E}[Y]$, $\mathbb{E}[Y^2]$, $\operatorname{Var} Y$, and the standard deviation σ_Y.

Remark: If the computation of $\mathbb{E}[Y^2]$ gets too tedious, just write how you will do it, without finishing the calculations.

(e) On average, on how many days of the year (at 6:01 p.m.) are there no newspapers on the pile? Explain how you obtained your result.

Problem 2. In Problem 3 of Homework 3 you considered a Markov chain with five states, and after relabeling the states the transition matrix of the Markov chain became

$$\mathbf{P} = \left(\begin{array}{c|c|c} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array}\right) = \left(\begin{array}{cc|c|cc} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{2} & 0 \end{array}\right);$$

here $\mathbf{0}$ denotes a matrix of appropriate size with zero entries, while a star denotes an arbitrary matrix of appropriate size. You proved that $\mathbf{C}_1$ and $\mathbf{C}_2$ are stochastic matrices, while $\mathbf{T}$ is not.

(a) Consider the closed and irreducible set C_1 comprising the recurrent states 1 and 2. Directly from the transition probabilities find the probabilities $\rho_{ij}^{(n)}$ of visiting state j for the first time in exactly n steps starting from state i for all possible i and j in C_1 (draw a simple diagram with these two states and think about the number of ways the first returns/visits can occur). Use the values you obtained to compute the probabilities f_{ij} of eventually visiting state j starting from state i for all i and j in C_1. Are you surprised by the results for f_{ij}? Explain why (or why not).

Remark: Here are some values that you can use without deriving:

$$\rho_{12}^{(n)} = \frac{2}{3^n} \text{ for all } n \in \mathbb{N} \;; \qquad \rho_{21}^{(1)} = 1 \;, \quad \rho_{21}^{(m)} = 0 \text{ for } m = 2, 3, 4, \ldots$$

(b) Find $\rho_{33}^{(n)}$ and f_{33} for the only state in the set C_2. Answer the same questions as in part (a).

(c) Compute the values $\rho_{54}^{(n)}$ and $\rho_{55}^{(n)}$, as well as f_{54} and f_{55}. Discuss your finding in the light of the general theory.

(d) Find the most general form of a stationary distribution $\boldsymbol{\pi}$ (satisfying $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ and the normalization condition). You will very easily see from the linear system that $\pi_4 = 0$ and $\pi_5 = 0$ (but you have to see this from the system!). How do you explain this fact?

(e) Finish the calculation of $\boldsymbol{\pi}$ started in part (d). How many stationary distributions does this Markov chain have (judging from concrete form of $\boldsymbol{\pi}$ you just obtained)? Discuss this in the light of the ergodic theorem.

(f) Can you suggest a method for computing all stationary distributions of the Markov chain in this problem without ever solving a system of five equations? Explain briefly how you are going to do it, and why your method will work.

Problem 3. This is a continuation of the previous problem. The form in which we wrote the transition probabilities $\mathbf{P}$ (after relabeling the states) is very convenient for studying the long-term behavior of the Markov chain because of the following fact (easy to check):

$$\mathbf{P}^n = \left(\begin{array}{c|c|c} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array}\right)^n = \left(\begin{array}{c|c|c} \mathbf{C}_1^n & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2^n & \mathbf{0} \\ \hline * & * & \mathbf{T}^n \end{array}\right)$$

(a) Consider only the irreducible matrix $\mathbf{C}_1$ containing only the recurrent states 1 and 2. Let μ_i be the average number of transitions needed by the process, starting from state i, to return to i for the first time (sometimes μ_i is called the *mean recurrence time* of state i). In part (a) of the previous problem you found the probabilities $\rho_{ii}^{(n)}$ of returning to state i from the initial state i for the first time in exactly n steps (for

$i \in \{1,2\}$). Use your results to compute μ_1 and μ_2. Are the states 1 and 2 positive recurrent or null recurrent? DId you expect what you just observed? Explain briefly.

Hint: The following trick is very useful for evaluating sums: differentiating with respect to q both sides of the formula for the sum of a geometric series,

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} , \qquad |q| < 1 ,$$

one obtains

$$\sum_{n=1}^{\infty} n q^{n-1} = \frac{1}{(1-q)^2} , \qquad |q| < 1$$

(in the sum in the left-hand side one can start the summation from 0, but the term with $n = 0$ is equal to zero). (Incidentally, differentiating one more time, one can obtain an expression for $\sum_{n=2}^{\infty} n(n-1)q^n$, from which $\sum_{n=2}^{\infty} n^2 q^n$ can be found, etc.)

(b) Consider only the matrix $\mathbf{C}_1$ containing the recurrent states 1 and 2. Since these two states form an irreducible set, the Ergodic Theorem guarantees that it has a unique stationary distribution $\widetilde{\boldsymbol{\pi}} = (\widetilde{\pi}_1 \ \widetilde{\pi}_2)$. Find $\widetilde{\boldsymbol{\pi}}$.

(c) Your results in parts (a) and (b) are related. How?

(d) In the rest of this problem you will consider the transient states 4 and 5. Recall that $I_A : S \to \{0,1\}$ is the indicator function of the event A:

$$I_A(s) = \begin{cases} 1 & \text{if } s \in A , \\ 0 & \text{if } s \notin A . \end{cases}$$

Let j be a transient state,

$$Y_j := \sum_{n=0}^{\infty} I_{\{X_n = j\}}$$

be the number of times the chain visits it, and

$$E[Y_j \,|\, X_0 = k] = E\left[\sum_{n=0}^{\infty} I_{\{X_n = j\}} \,\middle|\, X_0 = k \right]$$

be the expected number of times the chain visits the transient state j if initially it is in the transient state k. Show that

$$E[Y_j \,|\, X_0 = k] = \sum_{n=0}^{\infty} p_{kj}^{(n)} = \left(\sum_{n=0}^{\infty} \mathbf{P}^{(n)} \right)_{kj} = \left(\sum_{n=0}^{\infty} \mathbf{P}^{n} \right)_{kj} = \left((\mathbf{I} - \mathbf{T})^{-1} \right)_{kj} ,$$

where $\mathbf{I}$ is the unit matrix of appropriate size.

Hint: Use the fact that, for a matrix $\mathbf{A}$, if the "geometric series" $\sum_{n=0}^{\infty} \mathbf{A}^n$ converges, its sum is equal to

$$\sum_{n=0}^{\infty} \mathbf{A}^n = (\mathbf{I} - \mathbf{A})^{-1} .$$

Note that for numbers (i.e., 1×1 matrices) this becomes the well-known formula.

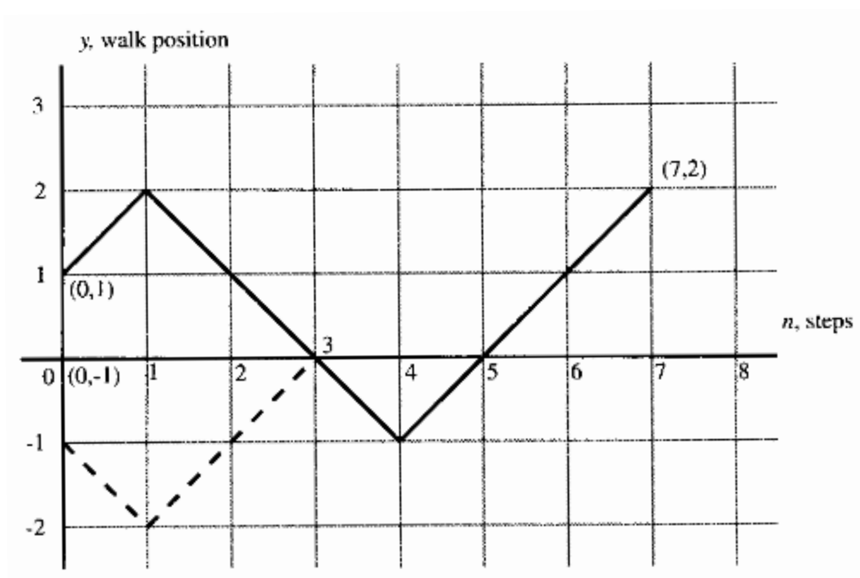
Remark: If $\mathbf{T}$ corresponds to the transient states only, there is a theorem that guarantees that the matrix $(\mathbf{I} - \mathbf{T})$ is invertible.

(g) I did the math, and obtained

$$\mathbf{T} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} , \qquad \mathbf{I} - \mathbf{T} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} , \qquad (\mathbf{I} - \mathbf{T})^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} .$$

Discuss the meaning of each entry of $(\mathbf{I} - \mathbf{T})^{-1}$ in the light of what you proved in part (f).

Problem 4. A random walk can be represented as a connected graph between coordinates (n, y), where the ordinate y is the position of the walk, and the abscissa n represents the number of steps. A walk of 7 steps which joins $(0, 1)$ and $(7, 2)$ is shown in the figure below. Suppose that a random walk starts at $(0, y_1)$ and finishes at (n, y_2), where $y_1 > 0$, $y_2 > 0$,



and $n + y_2 - y_1$ is an even number. Suppose also that the walk first visits the origin (i.e., position $y = 0$) at time $n = n_1$. Reflect that part of the path for which $n \leq n_1$ in the n-axis (see the figure), and use a reflection argument to show that the number of paths from $(0, y_1)$ to (n, y_2) which touch or cross the n-axis is equal to the number of *all* pats from $(0, -y_1)$ to (n, y_2). This is known as the *reflection principle.*