MATH 5483 Homework 4 Due Thu, Feb 14

Problem 1. Let $\{g_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence on $\mathbb{R}$, and $f \in L^2(\mathbb{R})$. Define the sequence of L^2 errors, $\{E_N\}_{N\in\mathbb{N}}$, by

$$E_N := \left\| f - \sum_{n=1}^{N} \langle f, g_n \rangle \, g_n \right\|_2 .$$

(a) Prove that the sequence $\{E_N\}_{N\in\mathbb{N}}$ is non-decreasing.

Hint: Use Lemma 2.50 or 2.51 from the book.

(b) Will the sequence $\{E_N\}_{N\in\mathbb{N}}$ always converge? Why?

(c) Under what condition on $\{g_n\}_{n\in\mathbb{N}}$ will the sequence $\{E_N\}_{N\in\mathbb{N}}$ converge to 0 as $N \to \infty$?

Problem 2. The *convolution*, $f * g$, of the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}} f(t)\, g(x-t)\, \mathrm{d}t ,$$

whenever the integral makes sense.

(a) Show that $f * g = g * f$.

(b) Prove that if $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g$ is also in $L^1(\mathbb{R})$ and

$$\|f * g\|_1 \le \|f\|_1 \, \|g\|_1 .$$

Hint: You can prove this directly:

$$\|f*g\|_1 = \int_{\mathbb{R}} |(f*g)(x)|\, \mathrm{d}x = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t)\, g(x-t)\, \mathrm{d}t \right| \mathrm{d}x \le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t)|\; |g(x-t)|\, \mathrm{d}t\, \mathrm{d}x ,$$

and then change the variable in the integral over x.

(c) Let $f \in L^1(\mathbb{R})$ and

$$\widehat{f}(\gamma) := \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i \gamma x}\, \mathrm{d}x$$

be its Fourier transform (which is well-defined for functions in $L^1(\mathbb{R})$). Show that

$$\widehat{f * g}(\gamma) = \widehat{f}(\gamma)\, \widehat{g}(\gamma) .$$

Problem 3. Let for any $\tau > 0$, the function $K_\tau : \mathbb{R} \to \mathbb{R}$ be defined as

$$K_\tau(x) = \frac{1}{\tau}\, \chi_{[-\tau/2,\tau/2)}(x) \ .$$

(a) Sketch the graph of K_τ.

(b) Prove the following properties of K_τ:

(i) for all $\tau > 0$, $\int_{\mathbb{R}} K_\tau(x)\, \mathrm{d}x = 1$;

(ii) there exists a constant $M > 0$ such that for all $\tau > 0$, $\int_{\mathbb{R}} |K_\tau(x)|\, \mathrm{d}x \leq M$;

(iii) for every $\delta > 0$,

$$\lim_{\tau \to 0+} \left[\int_{-\infty}^{-\delta} |K_\tau(x)|\, \mathrm{d}x + \int_{\delta}^{\infty} |K_\tau(x)|\, \mathrm{d}x \right] = 0 \ .$$

A function that satisfies these properties is called an *approximate identity* or a *summability kernel* on $\mathbb{R}$. (You can find more properties of this class of functions in Sec. 2.2 in the book, but you don't need them now.)

(c) Show that, if f is continuous on the compact interval $[a, b]$, then

$$\lim_{\tau \to 0+} (f * K_\tau)(x) = f(x) \ ,$$

(where the star stands for the convolution). State explicitly which properties and theorems you use (and where).

Hint: Use the Intermediate Value Theorem for a continuous function on a compact interval.

(d) If a function f has a finite discontinuity, then the convolution $f * K_\delta$ is continuous for any $\delta > 0$. Compute the convolution of K_δ with the *unit step function* (also called the *Heaviside function*)

$$\Theta(x) = \begin{cases} 0 & \text{if } x < 0 \ , \\ 1 & \text{if } x > 0 \ , \end{cases}$$

and sketch $\Theta * K_\delta$. Is $\Theta * K_\delta$ indeed continuous?