

**Note:** We adopt the following terminology: A function  $F : \mathbb{R} \rightarrow \mathbb{R}$  is *increasing* if  $x < y$  implies  $F(x) \leq F(y)$ . (Such functions are often called *non-decreasing*.)

**Problem 1.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing, right continuous function, and  $\mu_F$  be the corresponding Lebesgue-Stieltjes measure. Let  $F(a-) := \lim_{x \uparrow a} F(x)$  and  $F(a+) := \lim_{x \downarrow a} F(x)$  stand for the left and right limits, respectively.

- (a) Show that, for any  $a \in \mathbb{R}$ ,  $\mu_F(\{a\}) = F(a) - F(a-)$ . For a given  $a \in \mathbb{R}$ , what should  $F$  be so that  $\mu_F(\{a\}) = 0$ ?
- (b) Prove that  $\mu_F([a, b)) = F(b-) - F(a-)$ .
- (b) Give a proof that  $\mu_F([a, b]) = F(b) - F(a-)$ .
- (b) How can  $\mu_F((a, b))$  be written in terms of the values of the function  $F$ ?

**Problem 2.** Let  $A \subset [0, 1]$  be defined as

$$A := \left\{ x = 0.a_1a_2a_3 \dots := \sum_{j=1}^{\infty} \frac{a_j}{10^j} : a_n = 2 \text{ or } 7 \right\} .$$

Prove or disprove the following statements:

- (a)  $A$  is closed;
- (b)  $A$  is open;
- (c)  $A$  is countable;
- (d)  $A$  is dense in  $[0, 1]$ ;
- (e)  $A$  is Borel measurable.

**Problem 3.** Let the Lebesgue-Stieltjes measure  $\mu$  on  $\mathbb{R}$  be such that

$$\mu\left(\frac{1}{2^j}\right) = \frac{1}{2^j}, \quad j \in \mathbb{N},$$

and the measure of all intervals that do not contain any point of the form  $\frac{1}{2^j}$  is zero. Construct explicitly an increasing, right-continuous function  $F$  such that  $\mu = \mu_F$ . Plot the graph of  $F$ . What is  $\mu(\mathbb{R})$ ?

**Problem 4.** Let  $F : \mathbb{F} \rightarrow \mathbb{R}$  be an increasing function, and let

$$\mathcal{D}_F := \{x \in \mathbb{R} : F \text{ is not continuous at } x\}$$

be the set of its discontinuities. Prove that  $\mathcal{D}_F$  is countable.

*Hint:* One can easily see that  $F(x-) \leq F(x) \leq F(x+)$  (where we use the notations from Problem 1). For  $x \in \mathbb{R}$ , let  $J_x$  be the open (possibly empty) interval defined by

$$J_x := \begin{cases} (F(x-), F(x+)) & \text{if } F(x-) < F(x+) , \\ \emptyset & \text{if } F(x-) = F(x+) . \end{cases}$$

If an open interval is non-empty, it must contain a rational number. Use this to construct an injective map  $\mathcal{D}_F \rightarrow \mathbb{Q}$ .

**Problem 5.** Let  $X$  and  $Y$  be sets, and  $f : X \rightarrow Y$  be an arbitrary function.

- (a) Prove that  $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$ .
- (b) Show that  $f(E \cap F) \subset f(E) \cap f(F)$ .
- (c) Give an example showing that, in general,  $f(E \cap F)$  does not contain  $f(E) \cap f(F)$ .
- (d) Prove that  $f^{-1}(E^c) = (f^{-1}(E))^c$ .
- (e) Give examples showing that neither of the inclusions  $f(E^c) \subset (f(E))^c$  and  $f(E^c) \supset (f(E))^c$  is valid in general.

One can also show that  $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$  and  $f(E \cup F) = f(E) \cup f(F)$ , but you do not need to prove this here.