MATH 5453 Homework 4 Due Thu, Sep 25

Note: We adopt the following terminology: A function $F : \mathbb{R} \to \mathbb{R}$ is *increasing* if $x < y$ implies $F(x) \leq F(y)$. (Such functions are often called *non-decreasing.*)

Problem 1. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right continuous function, and μ_F be the corresponding Lebesgue-Stieltjes measure. Let $F(a-) := \lim\limits_{x\uparrow a} F(x)$ and $F(a+) := \lim\limits_{x\downarrow a} F(x)$ stand for the left and right limits, respectively.

(a) Show that, for any $a \in \mathbb{R}$, $\mu_F(\{a\}) = F(a) - F(a-)$. For a given $a \in \mathbb{R}$, what should F be so that $\mu_F(\{a\}) = 0$?

(b) Prove that $\mu_F([a, b)) = F(b-) - F(a-)$.

(b) Give a proof that $\mu_F([a, b]) = F(b) - F(a-)$.

(b) How can $\mu_F((a, b))$ be written in terms of the values of the function F?

Problem 2. Let $A \subset [0, 1]$ be defined as

$$A := \left\{ x = 0.a_1a_2a_3 \ldots := \sum_{j=1}^{\infty} \frac{a_j}{10^j} \;\; : \;\; a_n = 2 \text{ or } 7 \right\} .$$

Prove or disprove the following statements:

(a) A is closed;

(b) A is open;

(c) A is countable;

(d) A is dense in $[0, 1]$;

(e) A is Borel measurable.

Problem 3. Let the Lebesgue-Stieltjes measure μ on $\mathbb{R}$ be such that

$$\mu\left(\frac{1}{2^j}\right) = \frac{1}{2^j} , \quad j \in \mathbb{N} ,$$

and the measure of all intervals that do not contain any point of the form $\frac{1}{2^j}$ is zero. Construct explicitly an increasing, right-continuous function F such that $\mu = \mu_F$. Plot the graph of F. What is $\mu(\mathbb{R})$?

Problem 4. Let $F : \mathbb{F} \to \mathbb{R}$ be an increasing function, and let

$$\mathcal{D}_F := \{x \in \mathbb{R} : F \text{ is not continuous at } x\}$$

be the set of its discontinuities. Prove that $\mathcal{D}_F$ is countable.

Hint: One can easily see that $F(x-) \leq F(x) \leq F(x+)$ (where we use the notations from Problem 1). For $x \in \mathbb{R}$, let J_x be the open (possibly empty) interval defined by

$$J_x := \begin{cases} (F(x-), F(x+)) & \text{if } \ F(x-) < F(x+) \ , \\ \emptyset & \text{if } \ F(x-) = F(x+) \ . \end{cases}$$

If an open interval is non-empty, it must contain a rational number. Use this construct an injective map $\mathcal{D}_F \to \mathbb{Q}$.

Problem 5. Let X and Y be sets, and $f : X \to Y$ be an arbitrary function.

(a) Prove that $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.

(b) Show that $f(E \cap F) \subset f(E) \cap f(F)$.

(c) Give an example showing that, in general, $f(E \cap F)$ does not contain $f(E) \cap f(F)$.

(d) Prove that $f^{-1}(E^c) = \left(f^{-1}(E)\right)^c$.

(e) Give examples showing that neither of the inclusions $f(E^c) \subset \left(f(E)\right)^c$ and $f(E^c) \supset \left(f(E)\right)^c$ is valid in general.

One can also show that $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$ and $f(E \cup F) = f(E) \cup f(F)$, but you do not need to prove this here.