

**Problem 1. [Exponentiating a non-diagonalizable matrix]**

Throughout this problem,  $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ , where  $a$  is a nonzero real constant.

- (a) Compute explicitly  $A^2$ ,  $A^3$ ,  $A^4$ , guess what the general form of  $A^n$  is and prove that your guess is true by using induction.
- (b) Use your result from part (a) to compute a closed expression (i.e., without infinite sums, etc.) of  $e^{At}$ .
- (c) Consider the initial value problem  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = \mathbf{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix}$ . Write its solution  $\mathbf{x}(t)$  by using the explicit form of  $e^{At}$  obtained in part (b).

**Problem 2. [Eigenvectors; stable and unstable manifolds of a FP]**

Consider the linear system

$$\mathbf{x}' = \begin{pmatrix} 13 & -16 \\ 8 & -11 \end{pmatrix} \mathbf{x} =: A\mathbf{x} . \quad (1)$$

- (a) Write down the characteristic equation of the matrix  $A$  and find its eigenvalues.
- (b) Find the eigenvectors of the matrix  $A$ .
- (c) Write the general solution of the system (1) by using your results from parts (a) and (b).
- (d) Find the stable manifold  $W_0^s$  and the unstable manifold  $W_0^u$  of the fixed point  $\mathbf{0}$  of the system (1).

**Problem 3. [One-parameter family in the trace-determinant plane]**

Consider the one-parameter family of linear systems

$$\mathbf{x}' = \begin{pmatrix} a & -a \\ 1 & 0 \end{pmatrix} \mathbf{x} =: A\mathbf{x} . \quad (2)$$

- (a) Sketch the path traced out by the one-parameter family (2) of linear systems in the trace-determinant plane as the parameter  $a$  varies.
- (b) Discuss any bifurcations that occur along this path and compute the corresponding values of the parameter  $a$  at which bifurcations occur.

**Problem 4. [Two-parameter family in the trace-determinant plane]**

Sketch the analogue of the trace-determinant plane for the two-parameter family of linear systems,

$$\mathbf{x}' = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mathbf{x} =: A\mathbf{x} , \quad (3)$$

in the  $(a, b)$ -plane. That is, identify the regions in the  $(a, b)$ -plane where the system (3) where this system has similar phase portraits.

**Problem 5. [Linearization of a nonlinear system at a fixed point]**

Consider the nonlinear system

$$\begin{aligned} x_1' &= 1 - x_1 e^{x_2} \\ x_2' &= x_1 x_2 , \end{aligned} \quad (4)$$

i.e.,  $\mathbf{x}' = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$ , with  $f_1(\mathbf{x}) = 1 - x_1 e^{x_2}$ ,  $f_2(\mathbf{x}) = x_1 x_2$ . The system (4) has only one fixed point, namely,  $\mathbf{x}^* = (1, 0)$ .

(a) Find the linearized system

$$\mathbf{u}' = D\mathbf{f}(\mathbf{x}^*) \mathbf{u} ,$$

where  $D\mathbf{f}(\mathbf{x}^*)$  is a constant matrix with entries  $[D\mathbf{f}(\mathbf{x}^*)]_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}^*)$ , where  $\mathbf{x}^* = (1, 0)$ .

(b) Find the eigenvalues and the eigenvectors of the linearized system.

Some integral lines of the nonlinear system (4) are represented in Figures 1 and 2.

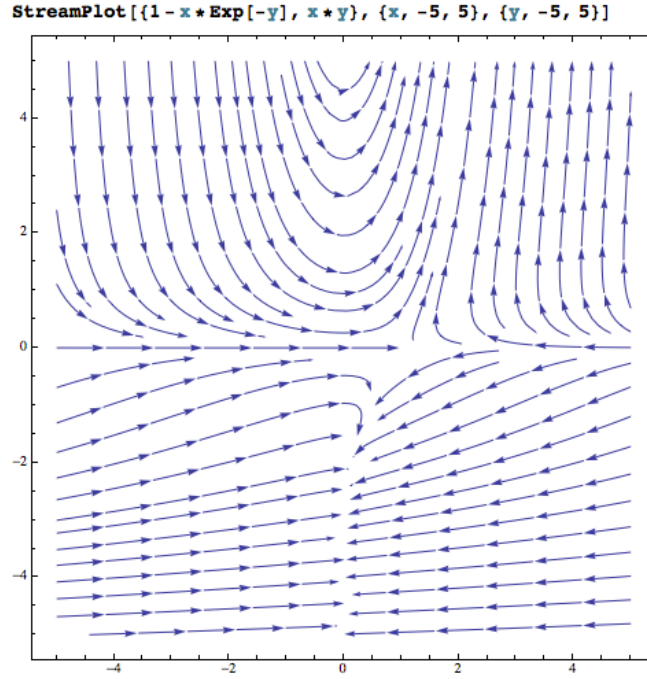


Figure 1: Integral lines of the nonlinear system (4) in the square  $[-5, 5] \times [-5, 5]$  (the fixed point of (4) is at  $\mathbf{x}^* = (1, 0)$ ).

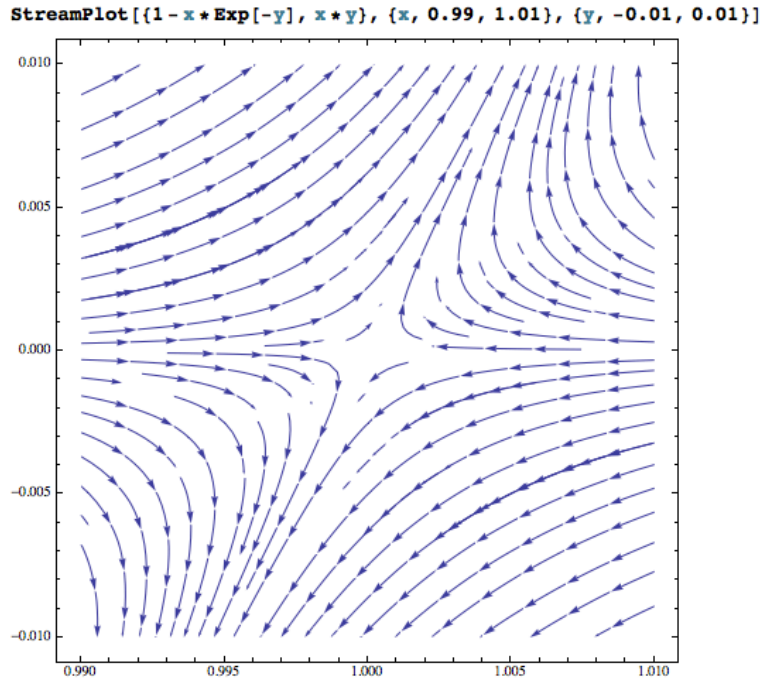


Figure 2: Integral lines of the nonlinear system (4) in a small square of side 0.02 centered at the fixed point  $\mathbf{x}^* = (1, 0)$ .