

Problem 1. [Exponentiating a non-diagonalizable matrix]

Throughout this problem, $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, where a is a nonzero real constant.

- Compute explicitly A^2 , A^3 , A^4 , guess what the general form of A^n is and prove that your guess is true by using induction.
- Use your result from part (a) to compute a closed expression (i.e., without infinite sums, etc.) of e^{At} .
- Consider the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix}$. Write its solution $\mathbf{x}(t)$ by using the explicit form of e^{At} obtained in part (b).

Problem 2. [Eigenvectors; stable and unstable manifolds of a FP]

Consider the linear system

$$\mathbf{x}' = \begin{pmatrix} 13 & -16 \\ 8 & -11 \end{pmatrix} \mathbf{x} =: A\mathbf{x} . \quad (1)$$

- Write down the characteristic equation of the matrix A and find its eigenvalues.
- Find the eigenvectors of the matrix A .
- Write the general solution of the system (1) by using your results from parts (a) and (b).
- Find the stable manifold W_0^s and the unstable manifold W_0^u of the fixed point $\mathbf{0}$ of the system (1).

Problem 3. [One-parameter family in the trace-determinant plane]

Consider the one-parameter family of linear systems

$$\mathbf{x}' = \begin{pmatrix} a & -a \\ 1 & 0 \end{pmatrix} \mathbf{x} =: A\mathbf{x} . \quad (2)$$

- Sketch the path traced out by the one-parameter family (2) of linear systems in the trace-determinant plane as the parameter a varies.
- Discuss any bifurcations that occur along this path and compute the corresponding values of the parameter a at which bifurcations occur.

Problem 4. [Two-parameter family in the trace-determinant plane]

Sketch the analogue of the trace-determinant plane for the two-parameter family of linear systems,

$$\mathbf{x}' = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mathbf{x} =: A\mathbf{x} , \quad (3)$$

in the (a, b) -plane. That is, identify the regions in the (a, b) -plane where the system (3) where this system has similar phase portraits.

Problem 5. [Linearization of a nonlinear system at a fixed point]

Consider the nonlinear system

$$\begin{aligned} x_1' &= 1 - x_1 e^{x_2} \\ x_2' &= x_1 x_2 , \end{aligned} \quad (4)$$

i.e., $\mathbf{x}' = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$, with $f_1(\mathbf{x}) = 1 - x_1 e^{x_2}$, $f_2(\mathbf{x}) = x_1 x_2$. The system (4) has only one fixed point, namely, $\mathbf{x}^* = (1, 0)$.

(a) Find the linearized system

$$\mathbf{u}' = D\mathbf{f}(\mathbf{x}^*) \mathbf{u} ,$$

where $D\mathbf{f}(\mathbf{x}^*)$ is a constant matrix with entries $[D\mathbf{f}(\mathbf{x}^*)]_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}^*)$, where $\mathbf{x}^* = (1, 0)$.

(b) Find the eigenvalues and the eigenvectors of the linearized system.

Some integral lines of the nonlinear system (4) are represented in Figures 1 and 2.

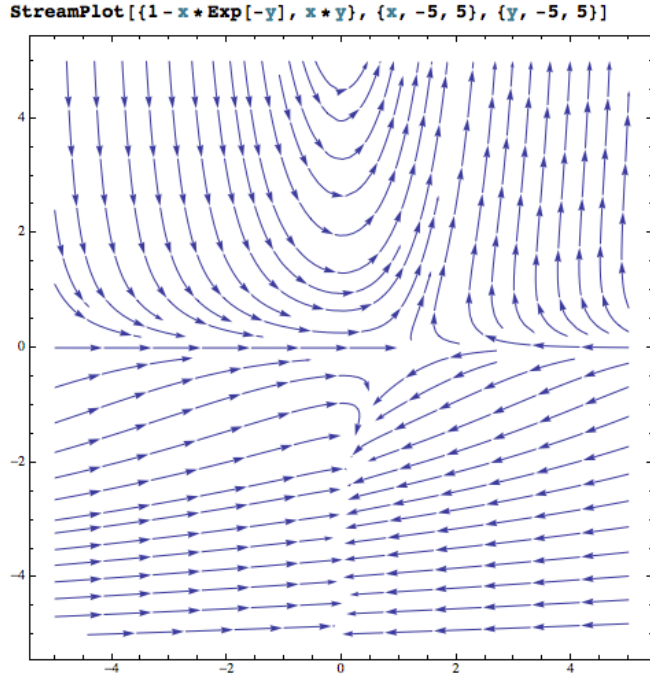


Figure 1: Integral lines of the nonlinear system (4) in the square $[-5, 5] \times [-5, 5]$ (the fixed point of (4) is at $\mathbf{x}^* = (1, 0)$).

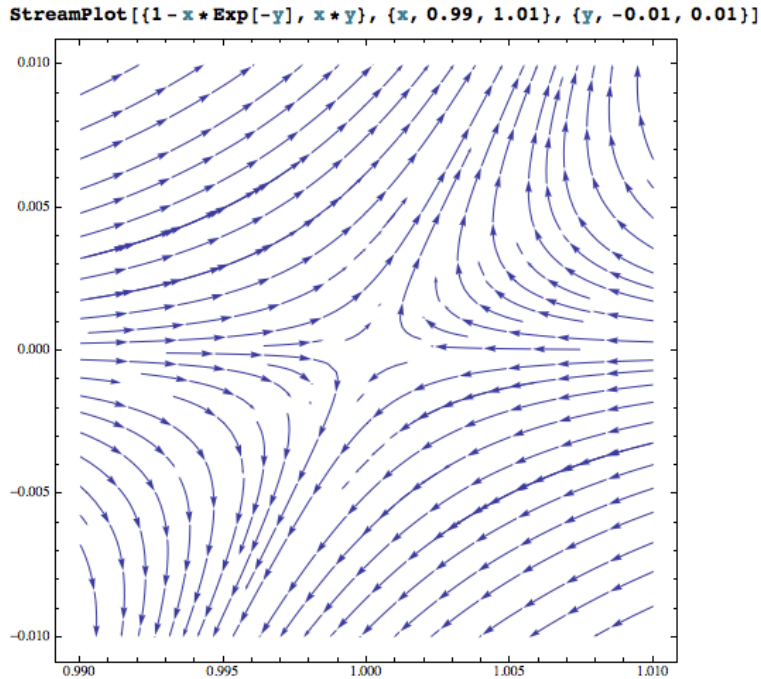


Figure 2: Integral lines of the nonlinear system (4) in a small square of side 0.02 centered at the fixed point $\mathbf{x}^* = (1, 0)$.