MATH 4433 **Homework 4** **Due 2/16/17 (Thursday)**

Abbott, Section 1.3: Exercises 1.3.6 (page 19).

Abbott, Section 1.4: Exercises 1.4.2, 1.4.4, 1.4.5, 1.4.8 (page 24).

Abbott, Section 1.5: Exercises 1.5.1, 1.5.4, 1.5.5 (page 30).

Abbott, Section 1.6: Exercise 1.6.4 (page 34).

Additional Problem 1. Find $\sup A$ and $\inf A$ of the set

$$A := \left\{ \frac{1}{n} - \frac{1}{m} \,:\, m, n \in \mathbb{N} \right\} \ ;$$

give a detailed proof of your claim.

Additional Problem 2. We start with a series of definitions and examples.

Definition. If A and B are sets, then their *Cartesian product* (or *cross product*, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$:

$$A \times B := \{(a, b) \,:\, a \in A,\, b \in B\} \ .$$

Definition. A *relation* F between a set A and a set B is any subset F of the Cartesian product $A \times B$. We say that an element $a \in A$ is *related* by F to an element $b \in B$ if $(a, b) \in \mathsf{F}$, which is often written as $a\mathsf{F}b$.

Definition. A *relation* F on a set A is any subset F of the Cartesian product $A \times A$. We say that an element $a \in A$ is *related* by F to an element $b \in A$ if $(a, b) \in \mathsf{F}$.

Example. As an example of a relation on $\mathbb{R}$, think of the plane $\mathbb{R}^2$ as a Cartesian product $\mathbb{R} \times \mathbb{R}$ – indeed, each point in the plane can be described as an ordered pair (x, y) of its x- and y-coordinates. Consider the set $\mathsf{F} \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ which consists of all points on and above the diagonal (the diagonal is defined as the straight line of slope 1 through the origin). Then a point $(x, y) \in \mathbb{R}^2$ belongs to F (written as $x\mathsf{F}y$) exactly when $x \leq y$.

Definition. A relation F on a set A is an *equivalence relation* if it has the following properties for all a, b, c in A:

- *reflexivity*: $a\mathsf{F}a$;
- *symmetry*: if $a\mathsf{F}b$, then $b\mathsf{F}a$;
- *transitivity*: if $a\mathsf{F}b$ and $b\mathsf{F}c$, then $a\mathsf{F}c$.

Examples.

1. The relation "$\leq$" on $\mathbb{N}$ is reflexive and transitive, but not symmetric, therefore it is not an equivalence relation.

2. When considering lines in the plane, the relation "is parallel to" is reflexive (if we agree that a line is parallel to itself), symmetric, and transitive. Hence it is an equivalence relation.

3. Let A be the set of all people who live in Norman, and suppose that two people, a and b, are related by F if a lives within a mile of b. Then F is reflexive and symmetric, but not transitive.

Given an equivalence relation F on a set A, it is natural to group together all the elements that are related to a particular element.

Definition. The *equivalence class* (with respect to F) of $a \in A$ is the set

$$E_a := \{b \in A \,:\, a\mathsf{F}b\} \;.$$

Since F is reflexive, each element of A is in some equivalence class. Furthermore, two different equivalence classes are disjoint, i.e., if two equivalence classes overlap, they must be equal (one can prove this by contradiction, using the symmetry and transitivity of F).

Here comes the problem.

(a) Consider the set $A = \{1, 2, 3\}$, and let F be the relation

$$\{(1,1),\ (1,2),\ (2,2),\ (1,3),\ (3,3)\} \subseteq A \times A \;.$$

Determine which of the three properties (reflexivity, symmetry, and transitivity) apply to this relation.

(b) Let $A = \mathbb{R}$, and F be the relation given by $a\mathsf{F}b$ if and only if $a - b$ is irrational. Determine which of the three properties (reflexivity, symmetry, and transitivity) apply to this relation.

(c) Let A be the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and define a relation F on A by $(a,b)\mathsf{F}(c,d)$ if and only if $a = c$. Verify that F is an equivalence relation and describe the equivalence class $E_{(7,3)}$ of the point $(7,3) \in \mathbb{R}^2$.