MATH 4193/5103 Homework 4 Due Thu, Apr 18, 2013

Problem 1. [Poincaré-Bendixson Theorem]
Consider the system

$$\dot{x} = x - y - x(x^2 + 5y^2) , \qquad \dot{y} = x + y - y(x^2 + y^2) . \tag{1}$$

(a) Classify the fixed point at the origin.

(b) Rewrite the system in polar coordinates, using that $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$.

Hint: The answer is $\dot{r} = r - r^3 - 4r^3\cos^2\theta\sin^2\theta$, $\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta$, but I want to see your calculations.

(c) Prove that the maximum value of the function $\varphi(\theta) := (\cos\theta\sin\theta)^2$ is $\frac{1}{4}$ (if θ is allowed to take any value). The easiest way to answer this question is to use some *very* elementary trigonometry. What is the minimum value that the function $\varphi(\theta)$ takes?

(d) Determine the circle of maximum radius, r_1, centered at the origin such that all trajectories have a radially outward component on it.

(e) Determine the circle of minimum radius, r_2, centered at the origin such that all trajectories have a radially inward component on it.

(f) Prove that the system (1) has a limit cycle in the trapping region $r_1 \le r \le r_2$. Figure 1 shows the result of numerical integration of the system (1). If you are taking the class as MATH 4193, you may assume without proof that there are no fixed points of the system (1) in the trapping region (this follows directly from part (g) which is only for those taking the class as 5103).

(g) **Only if you take the class as 5103!**

Show that there are no fixed points in the trapping region found in part (f).

Hint: One way to prove this is to exclude r from the system $\frac{\dot{r}}{r} = 0$, $\dot{\theta} = 0$, and then to show that the resulting equation for θ has no solution. To avoid long calculations, you may use a computer to plot some function of one variable.

Problem 2. [Nullclines and bifurcations]
In the article

P. Gray and S. Scott, Sustained oscillations and other exotic patterns of behavior in isothermal reactions, *Journal of Physical Chemistry*, Vol. 89 (1985), pp. 22–32

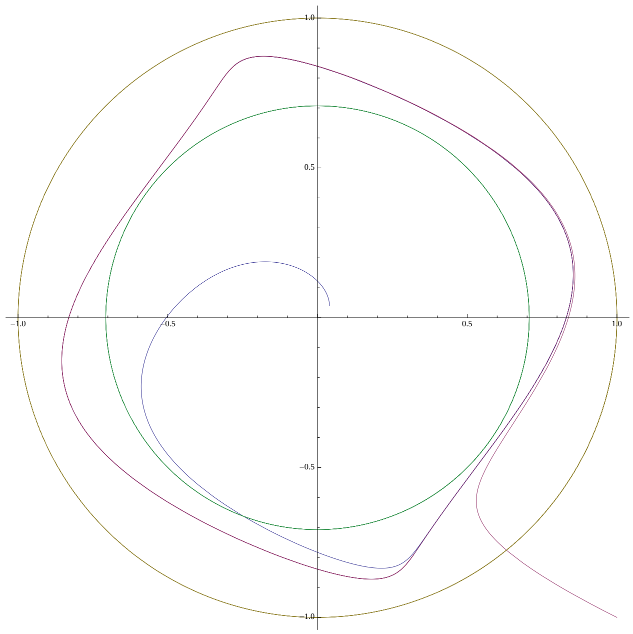


Figure 1: A limit cycle of the system (1), two phase trajectories, and the boundaries of the trapping region.

the authors consider a hypothetical isothermal autocatalytic reaction whose kinetics are given in dimensionless form by the two-parameter system

$$\begin{aligned} \dot{x} &= a(1-x) - xy^2 \ , \\ \dot{y} &= xy^2 - (a+b)y \ . \end{aligned} \tag{2}$$

Here $a > 0$ and $b > 0$ are positive parameters; the functions $x(t)$ and $y(t)$ may take any values in $\mathbb{R}$.

(a) Show that the equations of the nullclines can be written as follows:

$$\begin{aligned} (\dot{x} = 0)\text{-nullcline}: &\quad x = \varphi(y) := \frac{a}{a+y^2} \ , \\ (\dot{y} = 0)\text{-nullcline}: &\quad x = \psi(y) := \frac{a+b}{y} \quad \text{or} \quad y \equiv 0 \ . \end{aligned} \tag{3}$$

(b) Prove that at $b = -a \pm \frac{1}{2}\sqrt{a}$, the nullclines $\{x = \varphi(y)\}$ and $\{x = \psi(y)\}$ become tangent. Show that the coordinates $(x^*_\pm, y^*_\pm)$ of the points where the tangencies occur are $(\frac{1}{2}, \pm\sqrt{a})$. Note that the same sign (either $+$ or $-$) should be used in the expression for b and in the expression for $(x^*_\pm, y^*_\pm)$.

Remark: It is clear that the nullclines $\{x = \varphi(y)\}$ and $\{y \equiv 0\}$ can never be tangent, so you do not need to consider this.

(c) What is the significance of what you found in part (b)? Discuss its meaning from point of view of the bifurcations occurring in the systems, naming specifically the type of bifurcations that occurs.

(d) Take $a = 1$ and consider the $+$ sign in all expressions from part (b). We know from part (b) that when $b = -1 + \frac{1}{2}\sqrt{1} = -\frac{1}{2}$, the nullclines will be tangent at the point $(x_+^*, y_+^*) = (\frac{1}{2}, 1)$. Let us consider what happens if b is slightly off, i.e., take $b = -\frac{1}{2} + \xi$, where ξ is a very small number (positive or negative). For these values of a and b, the equations (3) of the nullclines become $x = \frac{1}{1+y^2}$, $x = \frac{\frac{1}{2}+\xi}{y}$. Show that from this system we can obtain the following quadratic equation for y:

$$y^2 - (\tfrac{1}{2} + \xi)^{-1}\, y + 1 = 0 \; .$$

Since ξ is very small, we have $\frac{1}{\frac{1}{2}+\xi} = \frac{2}{1-(-2\xi)} = 2\,[1 + (-2\xi) + (-2\xi)^2 + \cdots] \approx 2(1-2\xi)$, so in this approximation the quadratic equation for y becomes

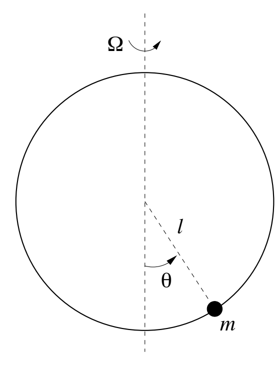
$$y^2 - 2(1 - 2\xi)\, y + 1 = 0 \; .$$

Show that, neglecting all high-order in ξ terms, we can write the solutions of this quadratic equation as $y_{1,2} = 1 - 2\xi \pm 2\sqrt{-\xi}$. Discuss why there are no solutions for $\xi > 0$ while there are two solutions for $\xi < 0$.

(e) Leaving only the term of lowest order with respect to ξ, we can write $y_{1,2} \approx 1 \pm 2\sqrt{-\xi}$. In the same approximation, find the corresponding values $x_{1,2}$. Please write your calculations in detail.

Problem 3. [A bead on a rotating hoop]

A bead of mass m can slide without friction on a circular hoop of radius ℓ that rotates about a vertical diameter with constant angular speed Ω as shown in the figure.



The equation of motion of the bead can be shown to be

$$m\ell\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = m\ell\,\Omega^2\cos\theta\sin\theta - mg\sin\theta \; , \tag{4}$$

where $\theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$; we think of S^1 as the interval $(-\pi, \pi]$ with identified ends. By introducing the dimensionless time $\tau := t\sqrt{\frac{g}{\ell}}$ and the non-negative dimensionless parameter $\mu := \frac{\ell\Omega}{g} \geq 0$, we can rewrite (4) as the system

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \nu \ , \qquad \frac{\mathrm{d}\nu}{\mathrm{d}\tau} = (\mu\cos\theta - 1)\sin\theta \ . \tag{5}$$

The parameter μ is the square of the ratio of the angular velocity Ω of the hoop's rotation and the frequency $\sqrt{\frac{g}{\ell}}$ of the small oscillations of the bead when the hoop is not rotating.

(a) Find all fixed points (i.e., equilibrium solutions) of the system (5). Show that, if $\mu \leq 1$, there are two equilibria, while for $\mu > 1$ there are four equilibria.

(b) Linearize (5) around the fixed point $(\pi, 0)$. What kind of fixed point is it? Is it hyperbolic?

Hint: If (5) is written as $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x} = \mathbf{f}(\mathbf{x})$, then $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \mu(\cos^2\theta - \sin^2\theta) - \cos\theta & 0 \end{pmatrix}$.

(c) In the case $\mu < 1$, linearize (5) around the fixed point $(0, 0)$, and show that $(0, 0)$ is a center (hence, non-hyperbolic). Find the period of the motion around this fixed point as a function of the parameter μ.

Hint: If $\lambda_{1,2}$ are the eigenvalues of the matrix of the linearized system (recall that λ_1 is the complex conjugate of λ_2), then in the case of a center the period of the small periodic motions around the corresponding fixed point is $\frac{2\pi}{\operatorname{Im}\lambda}$.

(d) In the case $\mu > 1$, linearize (5) around the fixed point $(0, 0)$. What kind of fixed point is $(0, 0)$ in this case? Is it hyperbolic? Find its eigenvalues and eigenvectors.

(e) In the case $\mu > 1$, linearize (5) around the fixed point $(\arccos\frac{1}{\mu}, 0)$ and show that it is a center. Find the period of the motion around this fixed point as a function of the parameter μ.

(f) Use your results from (d) and (e) to sketch the phase portrait of the system in the case $\mu > 1$.

Remark: The behavior of the system around the fourth fixed point, $(-\arccos\frac{1}{\mu}, 0)$ is the same as around $(\arccos\frac{1}{\mu}, 0)$.

(g) Sketch the position of the four equilibria as functions of μ (use solid line for the stable equilibria and dashed line for the unstable ones). Find the positions of the four equilibria in the limit $\mu \to \infty$. What is the physical explanation of your result (in particular, in the limit $\mu \to \infty$)?

(h) What is the physical explanation of the bifurcation occurring at $\mu = 1$?

Problem 4. $[(P_y = Q_x) \not\Rightarrow (P\mathbf{i} + Q\mathbf{j} = \nabla f)]$ $\boxed{\textbf{Only if you take the class as 5103!}}$
Consider the vector field

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} := -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}\,, \qquad (x,y) \neq (0,0)\,. \tag{6}$$

(a) Show that $P_y(x,y) = Q_x(x,y)$ for any $(x,y) \neq (0,0)$.

(b) Let (r,θ) be the polar coordinates in $\mathbb{R}^2$, and $D := \mathbb{R}^2 \setminus \{(0,0)\}$ be the plane with its origin removed. Define the function

$$\Theta : D \to \mathbb{R}$$

as $\Theta(x,y) = \theta$; in other words, $\Theta(x,y)$ is the angle between the positive direction on the x-axis and the line connecting $(0,0)$ and (x,y), measured counterclockwise. For simplicity, consider only the function Θ in the right half-plane, i.e., in the set $\{(x,y) \in \mathbb{R}^2 : x > 0\}$, where Θ is given by

$$\Theta(x,y) = \tan^{-1}\frac{y}{x}\,.$$

Show that $\mathbf{F}(x,y) = \nabla\Theta(x,y)$ in the right half-plane.

Remark: You do not have to do this here, but one can define similarly the functions

$$\begin{aligned}
\Theta_1 &: \{(x,y) \in \mathbb{R}^2 : y > 0\} \to \mathbb{R} : (x,y) \mapsto \cot^{-1}\frac{x}{y}\,,\\
\Theta_2 &: \{(x,y) \in \mathbb{R}^2 : x < 0\} \to \mathbb{R} : (x,y) \mapsto \pi + \tan^{-1}\frac{y}{x}\,,\\
\Theta_3 &: \{(x,y) \in \mathbb{R}^2 : y < 0\} \to \mathbb{R} : (x,y) \mapsto \pi + \cot^{-1}\frac{x}{y}\,,
\end{aligned}$$

and in each of the corresponding domains, $\mathbf{F}(x,y) = \nabla\Theta_j(x,y)$.

(c) Let C be the unit circle in $\mathbb{R}^2$, with counterclockwise orientation. Compute the value of the line integral $\oint_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$.

Hint: You can parameterize the unit circle as follows: $\mathbf{R}(t) = (X(t), Y(t)) = (\cos t, \sin t)$, where $t \in [0, 2\pi)$.

(d) Explain why your result from part (c) implies that $\mathbf{F}$ cannot be a gradient of a scalar function defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.