MATH 4193/5103 **Homework 4** **Due 3/24/20 (Tuesday)**

Problem 1. [The golden mean]

In this problem you will find the exact value of the number γ, often called the *golden mean* or the *golden ratio* (sometimes this terminology is used for γ^{-1}). The golden mean is defined by the following expression:

$$\gamma = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}} \; . \tag{1}$$

(a) Consider the iteration $x_{n+1} = f(x_n)$, where $x_1 = 1$, and

$$f(x) = \frac{1}{1+x} \; .$$

Recall the following result.

Theorem.

(i) If the function $g : [a,b] \to \mathbb{R}$ is continuous and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.

(ii) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with $|g'(x)| \le k$ for all $x \in (a,b)$, then there is exactly one fixed point in $[a,b]$.

Find an interval $[a,b]$ that is mapped into itself by f and use this theorem to prove the existence of a fixed point of f in $[a,b]$.

(b) Use the above theorem to prove the uniqueness of the fixed point of f.

(c) Is the fixed point of f stable or unstable?

Hint: Recall Lecture 4.

(d) Sketch (by hand) a cobweb plot for the function f to illustrate how the iterates of f behave.

(e) Find the exact value of the fixed point of f.

(f) How is your result in part (e) related to the value of the golden mean? Explain briefly.

Problem 2. [Linearization of a nonlinear system at a hyperbolic fixed point]

Consider the nonlinear system

$$\begin{aligned} x_1' &= 1 - x_1 e^{x_2} \\ x_2' &= x_1 x_2 \; , \end{aligned} \tag{2}$$

i.e., $\mathbf{x}' = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$, with $f_1(\mathbf{x}) = 1 - x_1 e^{x_2}$, $f_2(\mathbf{x}) = x_1 x_2$. The system (2) has only one fixed point, namely, $\mathbf{x}^* = (1, 0)$.

(a) Let us first linearize the nonlinear system (2) "by hand". Define new vector-valued function $\mathbf{u}(t) = (u_1(t), u_2(t))$ by

$$\mathbf{x}(t) = \mathbf{x}^* + \mathbf{u}(t) \ , \tag{3}$$

i.e., set $x_1(t) = x_1^* + u_1(t)$, $x_2(t) = x_2^* + u_2(t)$. The vector-valued function $\mathbf{u}(t)$ is the displacement from the fixed point. Since we are interested in the behavior of the integral lines in a small neighborhood of the fixed point, we think of $\mathbf{u}(t)$ as small. In more detail, this means the following. Think of the functions $u_1(t)$ and $u_2(t)$ as having values so small that if one substitutes $\mathbf{x}(t)$ expressed in terms of the function $\mathbf{u}(t)$ into the right-hand side of the nonlinear system (2) and expands all the terms in the right-hand side of (2) in Taylor series, then one can leave only the terms that are linear in $u_1(t)$ and $u_2(t)$. In other words, ignore the quadratic terms $u_1(t)^2$, $u_1(t)u_2(t)$, $u_2(t)^2$, the cubic terms, and all terms of higher order. Then the nonlinear system (2) for the unknown functions $x_1(t)$, $x_2(t)$ becomes a linear constant-coefficient system for the new unknown functions $u_1(t)$, $u_2(t)$ – this is exactly the linearization of the original system (2) at the fixed point $\mathbf{x}^*$. Your goal in this part of the problem is only to write the linear constant-coefficient system for $u_1(t)$ and $u_2(t)$ by using this method. The resulting system is the linearization of the original nonlinear system (2) at the fixed point $\mathbf{x}^*$.

(b) Write down the linearized system

$$\mathbf{u}' = D\mathbf{f}(\mathbf{x}^*)\,\mathbf{u} \ ,$$

in the "standard" way (you will obtain the same result as in part (a)). Here $D\mathbf{f}(\mathbf{x}^*)$ is the Jacobian matrix at $\mathbf{x}^*$, i.e., a constant matrix with entries

$$[D\mathbf{f}(\mathbf{x}^*)]_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}^*) \ , \quad \text{where } \mathbf{x}^* = (1, 0) \ .$$

(c) Find the eigenvalues of the linearized system. Is the fixed point $\mathbf{x}^* = (1, 0)$ hyperbolic? What does this imply about the behavior of the linearized system compared with the behavior of the original system (2) in a small neighborhood of $\mathbf{x}^*$?

(d) Find the eigenvectors of the linearized system. Sketch the integral lines of the linearized system in a small neighborhood of the fixed point.

Some integral lines of the nonlinear system (2) are represented in Figure 1.

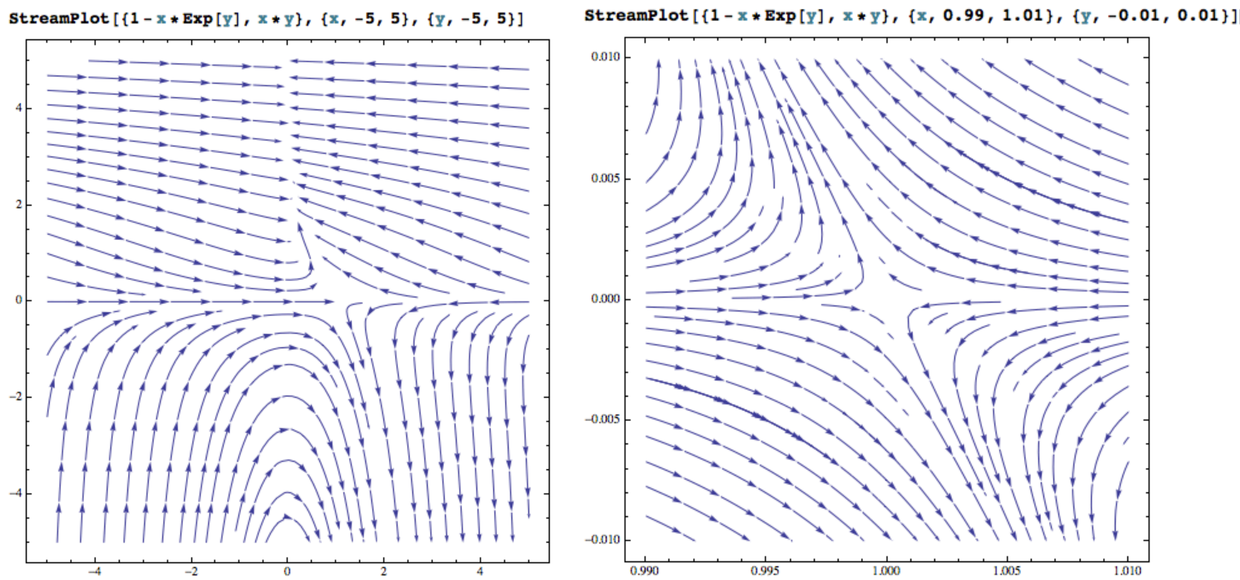

Figure 1: Integral lines of the nonlinear system (2) in the square $[-5, 5] \times [-5, 5]$ (left) and in the square $[-0.01, 0.01] \times [-0.01, 0.01]$ (right); the fixed point of (2) is at $\mathbf{x}^* = (1, 0)$.

Problem 3. [Linearization of a nonlinear system at a non-hyperbolic fixed point] Consider the nonlinear system

$$\begin{aligned} x' &= -y + \mu x(x^2 + y^2) \\ y' &= x + \mu y(x^2 + y^2) \ , \end{aligned} \tag{4}$$

where μ is a parameter. Obviously, the origin $\mathbf{x}^* = (0, 0)$ is a fixed point of (4).

(a) Linearize the non-linear system (4) at the fixed point $\mathbf{x}^* = (0, 0)$, i.e., write down the system

$$\mathbf{u}' = A\,\mathbf{u} \ , \quad A = D\mathbf{f}(\mathbf{x}^*) \ ,$$

where $\mathbf{u}(t) = (u(t), v(t))$ are the small displacements from the fixed point.

(b) What are the eigenvalues of the matrix A from part (a)? Is the fixed point $(0, 0)$ hyperbolic? Justify your answer.

(c) Solve the linearized system of ODEs derived in part (a). Perhaps the simplest way to solve it is to write down a second-order ODE for the first component, $u(t)$, of the vector-valued function $\mathbf{u}(t)$ – if you do that, you will immediate recognize the equation and can write its solution right away. Sketch the typical phase trajectories of the linearized system in the (u, v)-plane (which is the same as the (x, y)-plane).

(d) Now you will solve the nonlinear system (4). Perform a polar change of coordinates, i.e., introduce new pair of unknown functions, $r(t)$ and $\theta(t)$, related with $x(t)$ and $y(t)$

by

$$x(t) = r(t)\, \cos\theta(t)\ , \quad y(t) = r(t)\, \sin\theta(t)\ .$$

Performed detailed calculations to show that the nonlinear system (4) becomes

$$\begin{aligned} r' &= \mu r^3 \\ \theta' &= 1\ . \end{aligned}$$

(e) The solution of the ODE for $\theta(t)$ is obvious – the angle θ increases at a constant rate. Without solving the ODE for $r(t)$, explain how $r(t)$ behaves when $t \to \infty$ in the cases $\mu < 0$, $\mu = 0$, and $\mu > 0$. Sketch the typical phase trajectories in the (x, y)-plane in each of these three cases.

(f) Does the linearized system faithfully reflect the behavior of the nonlinear system (4) in a small neighborhood of the fixed point $(0, 0)$? Discuss your findings in the light of the Hartman-Grobman Theorem.

Problem 4. [One-parameter family in the trace-determinant plane]
Consider the one-parameter family of linear systems

$$\mathbf{x}' = \begin{pmatrix} a & -a \\ 1 & 0 \end{pmatrix} \mathbf{x} =: A\mathbf{x}\ . \tag{5}$$

(a) Sketch the path traced out by the one-parameter family (5) of linear systems in the trace-determinant plane as the parameter a varies. Indicate the important values in your picture.

(b) What can you say about the eigenvalues of A for $a \in (-\infty, 0)$? What is the type of the fixed point $(0, 0)$ of (5) when the parameter a is in this range? Sketch the phase portrait in this case.

(c) What can you say about the eigenvalues of A for $a \in (0, 4)$? Classify the type of the fixed point $(0, 0)$ of (5) and sketch the phase portrait in this case.

(d) Do the same as in parts (b) and (c) for $a \in (4, \infty)$.