MATH 4163 **Homework 4** **Due Thu, 02/15/2018**

Problem 1. [Uniqueness of solutions of heat equation with Neumann/Robin BCs]
Consider the initial-boundary value problem (IBVP) for the diffusion heat equation for the function $u(x,t)$ on the spatial interval $[0,L]$, with a source term $g(x,t)$, Neumann BC at the left end and Robin BC on the right end:

$$\begin{aligned} & u_t = \alpha^2 u_{xx} + g(x,t) \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\ & u_x(0,t) = a(t) \ , \\ & u_x(L,t) + \beta u(L,t) = b(t) \ , \\ & u(x,0) = f(x) \ . \end{aligned} \tag{1}$$

Assume that α and β are positive constants, and the functions g, a, b, and f are C^2.

(a) Let $u_1(x,t)$ and $u_2(x,t)$ be two C^2 solutions of the IBVP (1), and introduce the function $v : [0,L] \times \mathbb{R}_+ \to \mathbb{R}$ as their difference:

$$v(x,t) := u_2(x,t) - u_1(x,t) \ .$$

Show that the function v satisfies the IBVP

$$\begin{aligned} & v_t = \alpha^2 v_{xx} \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\ & v_x(0,t) = 0 \ , \\ & v_x(L,t) + \beta v(L,t) = 0 \ , \\ & v(x,0) = 0 \ . \end{aligned} \tag{2}$$

(b) Define the function $F : [0,\infty) \to \mathbb{R}$ by

$$F(t) := \int_0^L [v(x,t)]^2 \, \mathrm{d}x \ .$$

Explain why $F(0) = 0$ and $F(t) \geq 0$ for any $t \in [0,\infty)$.

(c) Differentiate $F(t)$ with respect to t and integrate by parts to show that

$$F'(t) = 2\alpha^2 \left[v(x,t) v_x(x,t) \Big|_{x=0}^{L} - \int_0^L [v_x(x,t)]^2 \, \mathrm{d}x \right] \ .$$

(d) Use the BCs for $v(x,t)$ to show that $F'(t) \leq 0$. Please write your calculations and explain your reasoning in detail.

(e) Explain clearly how your results from parts (b) and (d) imply that the solution of the IBVP (1) is unique.

Problem 2. [Solutions obtained by separation of variables]

(a) Use the method of separation of variables to write down the solution of the following Dirichlet IBVP for the heat equation:

$$\begin{aligned} &u_t = 25\, u_{xx} \ , \qquad (x,t) \in [0,3] \times \mathbb{R}_+ \ , \\ &u(0,t) = 0 \ , \\ &u(3,t) = 0 \ , \\ &u(x,0) = 7 \sin \frac{\pi x}{3} - 11 \sin(4\pi x) \ . \end{aligned} \tag{3}$$

You may use Proposition 1 on page 129 of Bleecker's book without rederiving it, just apply the result in the Proposition to this particular problem.

(b) Use the method of separation of variables to write down the solution of the following Dirichlet IBVP for the heat equation:

$$\begin{aligned} &u_t = 25\, u_{xx} \ , \qquad (x,t) \in [0,3] \times \mathbb{R}_+ \ , \\ &u(0,t) = 0 \ , \\ &u(3,t) = 0 \ , \\ &u(x,0) = 8 \sin \frac{11\pi x}{6} \cos \frac{\pi x}{2} \ . \end{aligned} \tag{4}$$

Again, you may use Proposition 1 on page 129 of Bleecker directly. You will need to write the initial condition, $f(x) = 8 \sin \frac{11\pi x}{6} \cos \frac{\pi x}{2}$, as a sum of sines. To this end, you will need a bit of trigonometry which you can find in some book, or can derive by using the following versatile trigonometric identities:

$$\begin{aligned} \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \ , \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \ . \end{aligned} \tag{5}$$

To express a product of a sine and a cosine (as in the IC of (6)) as a sum of sines, you may take the first identity from (5) with a plus and separately with a minus sign,

$$\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \ , \\ \sin(x - y) &= \sin x \cos y - \cos x \sin y \ , \end{aligned}$$

and add them to obtain

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y \ ,$$

which is exactly what is needed.

(c) Use the method of separation of variables to write down the solution of the following Neumann IBVP for the heat equation:

$$\begin{aligned} & u_t = 25\, u_{xx} \ , \qquad (x,t) \in [0,3] \times \mathbb{R}_+ \ , \\ & u_x(0,t) = 0 \ , \\ & u_x(3,t) = 0 \ , \\ & u(x,0) = 5 - 2\cos\frac{\pi x}{3} + \cos\frac{11\pi x}{3} \ . \end{aligned} \tag{6}$$

We discussed the solution of the heat equation with Neumann BCs in class, so you may take the general form the solution from there, without rederiving it.

Problem 3. [Separation of variables in heat equation for Dirichlet/Robin BCs] Consider the following IBVP with Dirichlet BCs on the left end and Robin BCs on the right end:

$$\begin{aligned} & u_t = \alpha^2 u_{xx} \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\ & u(0,t) = 0 \ , \\ & u_x(L,t) + \beta u(L,t) = 0 \ , \\ & u(x,0) = f(x) \ , \end{aligned} \tag{7}$$

where α and β are positive constants, and f is a C^2 function.

(a) Look for a solution of (7) of the form

$$u(x,t) = X(x)T(t) \ .$$

Plug this function in the PDE and derive ODEs that the functions $T(t)$ and $X(x)$ must satisfy.

(b) Show that the function $X(x)$ must satisfy the BVP

$$\begin{aligned} & X''(x) - \mu X(x) = 0 \ , \qquad x \in [0,L] \ , \\ & X(0) = 0 \ , \\ & X'(L) + \beta X(L) = 0 \ . \end{aligned} \tag{8}$$

(c) Assume that $\mu > 0$, set $\mu = \lambda^2$ for some $\lambda > 0$, and show that the BVP (8) has no non-trivial solutions.

(d) Solve the BVP (8) in the case $\mu = 0$ and show that it has no no non-trivial solution.

(e) Assume that $\mu < 0$ and set $\mu = -\lambda^2$ for some $\lambda > 0$. Find the general solution of the ODE for X.

(f) Impose the boundary conditions on X on the general solution for X found in part (e) to show that $X(x) = \sin \lambda x$, where the constant λ (equal to $\sqrt{-\mu}$) satisfies the transcendental equation

$$\tan(\lambda L) = -\tfrac{\lambda}{\beta}$$

or, equivalently, that λ must take values $\lambda = \frac{\xi}{L}$, where ξ is a solution of the equation

$$\tan \xi = -\tfrac{\xi}{\beta L} \ . \tag{9}$$

(g) Clearly, $\xi = 0$ is a solution of (9), but this would imply that $\lambda = 0$ which corresponds to a trivial solution as we found in part (d). Since $\tan \xi$ is an odd function, if some value of ξ satisfies (9), then $-\xi$ will also satisfy in. But $\sin(-\lambda x) = -\sin(\lambda x)$, so that we should only be interested in positive values of λ, so that we should look for only positive solutions $\xi > 0$ of (9). Assume that $\beta L > 0$ has certain value, and sketch in the same graph the graphs of the functions $\phi(\xi) = \tan \xi$ and $\phi(\xi) = -\frac{\xi}{\beta L}$ (the function $\tan \xi$ is periodic with period π). From your picture it will be clear that the transcendental equation (9) has infinitely many positive roots ξ_n, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and that they are all simple.

Remark: We say that the root $\xi^* \in \mathbb{R}$ of the equation $F(\xi) = 0$ is *simple* if $F(\xi^*) = 0$ and $F'(\xi^*) \neq 0$.

(h) If the roots of (9) are $\xi_1 < \xi_2 < \xi_3 < \cdots$, write down the corresponding values of λ_n and the corresponding solutions $X_n(x)$ of (8).

(i) Write down the solution of the ODE for the function $T_n(t)$ that correspond to ξ_n (resp., λ_n and μ_n).

Problem 4. [Eliminating constants by changing units]

Consider the heat equation, $u_t = \alpha^2 u_{xx}$ for the function $u(x, t)$ on the interval $x \in [0, L]$. It would be convenient if the constant α was equal to 1 (in the sense that the expressions we would write will look a bit simpler). If one solves this equation by separation of variables, it would also be convenient if L had value 1 or π.

One can make α equal to 1 and L equal to, say, π, by changing the units in which x and t are measured. The idea is the following: define new dimensionless variables $\widetilde{x}$ and $\widetilde{t}$, as follows:

$$\widetilde{x} = \tfrac{\pi}{L}\, x \ , \quad \widetilde{t} = \tfrac{t}{\sigma} \ ,$$

where σ is a constant that is measured in seconds. Define the new function $\widetilde{u}(\widetilde{x}, \widetilde{t})$ as usual.

(a) What is the range of the new variable $\widetilde{x}$?

(b) Write down the PDE satisfied by the new function $\widetilde{u}(\widetilde{x}, \widetilde{t})$. Choose the constant σ such that the new equation become $\widetilde{u}_{\widetilde{t}} = \widetilde{u}_{\widetilde{x}\widetilde{x}}$.

(c) If $\Phi(\widetilde{x}, \widetilde{t})$ is the solution of the PDE for $\widetilde{u}$, then what is the corresponding solution for the original function $u(x, t)$?