MATH 4163 **Homework 4** **Due Fri, Sep 21, 2012**

Problem 1. Solve the following BVP for the Laplace's equation:

$$\begin{aligned}
&\Delta u(x,y) = 0 \ , \qquad x \in [0,a] \ , \quad y \in [0,b] \ , \\
&u_x(0,y) = 0 \ , \quad u_x(a,y) = 0 \ , \\
&u(x,0) = 0 \ , \quad u(x,b) = f(x) \ ,
\end{aligned} \tag{1}$$

where the function f has Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{a} \ , \qquad a_j = \frac{2}{a}\int_0^a f(x)\,\cos\frac{j\pi x}{a}\,\mathrm{d}x \ , \quad j \in \{0,1,2,\ldots\} \ .$$

The physical meaning of the BVP (1) is the following: $u(x,y)$ can be interpreted as the steady-state temperature distribution in the rectangle $[0,a]\times[0,b]$ such that:

- there are no heat sources inside the rectangle $[0,a]\times[0,b]$;
- the walls at $x=0$ and $x=a$ are thermally insulated (i.e., the heat flux through each point of these walls is zero);
- the temperatures at the wall at $y=0$ is kept equal to zero, while the temperature at the wall at $y=b$ is given by the function $f(x)$.

You may use that the BVP

$$\begin{aligned}
&X''(x) - \mu X(x) = 0 \ , \qquad x \in [0,a] \ , \\
&X'(0) = 0 \ , \quad X'(a) = 0
\end{aligned}$$

has a non-zero solution only if μ takes one of the values

$$\mu_n = -\left(\frac{n\pi}{a}\right)^2 \ , \qquad n \in \{0,1,2,\ldots\} \ .$$

The corresponding solutions of this BVP are

$$X_n(x) = \begin{cases} 1 & \text{if } n = 0 \ , \\ \cos\dfrac{n\pi x}{a} & \text{if } n \in \mathbb{N} \ . \end{cases} \tag{2}$$

When you separate variables, the solution $u(x,y)$ of the BVP (1) is a superposition of functions of the form

$$u_n(x,y) = X_n(x)\,Y_n(y) \ , \tag{3}$$

where the functions X_n are given by (2).

(a) What ODEs do the functions Y_n in (3) satisfy? (Derive the ODEs separately for the cases $n = 0$ and $n \in \mathbb{N}$.)

The general solutions of the ODEs you just wrote are

$$Y_n(y) = \begin{cases} A_0 + B_0 y & \text{if } n = 0\ , \\ A_n \cosh \dfrac{n\pi y}{a} + B_n \sinh \dfrac{n\pi y}{a} & \text{if } n \in \mathbb{N} \end{cases} \tag{4}$$

(*you do not need to derive this!*). We could have written the solution for $Y_n(y)$ for $n \in \mathbb{N}$ as a sum of two exponents, but the representation in (4) is more convenient in this problem.

(b) Do the functions $u_n(x, y) = X_n(x)\, Y_n(y)$ satisfy the PDE from the BVP (1)? Why? Which of the four BCs does each of these functions satisfy?

(c) Write the expansion

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = \sum_{n=0}^{\infty} X_n(x)\, Y_n(y)\ ,$$

with the explicit expressions for X_n and Y_n. Impose the remaining BCs from the BVP (1) to find the constants in the functions Y_n. Write down the solution $u(x, y)$ of the BVP (1).

(d) Solve the BVP (1) in the case

$$f(x) = 5 + 3 \cos \frac{7\pi x}{a}\ .$$

Problem 2. In this problem you will attempt to solve the following BVP for the Laplace's equation:

$$\begin{aligned} &\Delta u(x, y) = 0\ , \qquad x \in [0, a]\ , \quad y \in [0, b]\ , \\ &u_x(0, y) = 0\ , \quad u_x(a, y) = 0\ , \\ &u_y(x, 0) = 0\ , \quad u_y(x, b) = f(x)\ , \end{aligned} \tag{5}$$

where the function f has Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{a}\ , \qquad a_j = \frac{2}{a} \int_0^a f(x) \cos \frac{j\pi x}{a}\, \mathrm{d}x\ , \quad j \in \{0, 1, 2, \ldots\}\ . \tag{6}$$

The BVP (5) is similar to the BVP (1) considered in Problem 1, but here the BCs on *all* walls are Neumann BCs (i.e., the derivative of $u(x, y)$ in a direction normal to the wall is given), while in Problem 1 the BCs on two walls were Neumann and on the other two walls the BCs were of Dirichlet type (i.e., the value of $u(x,y)$ at the wall is given). This difference seems very small, but in fact is crucial because of the physical interpretation of the BVPs (1) and (5).

The solution of this problem is very similar to the one of Problem 1 (some things are totally identical), so use your results from Problem 1 without rederiving them here. Assume that, again, we look for the solution $u(x,y)$ as a superposition of functions $u_n(x,y) = X_n(x)\,Y_n(y)$. The functions X_n are again given by (2). For the rest of the problem, follow the steps below.

(a) What ODEs do the functions Y_n satisfy? What are the general solutions of these ODEs?

Hint: How is this part of the problem different from part (a) of Problem 1?

(b) Do the functions $u_n(x,y) = X_n(x)\,Y_n(y)$ satisfy the PDE from the BVP (5)? Which of the four BCs does each of these functions satisfy?

(c) As in Problem 1(c), write the expansion

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) = A_0 + B_0 y + \sum_{n=1}^{\infty} \Big(A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a} \Big) \cos\frac{n\pi x}{a}$$

and impose the remaining BCs in (5) to derive equations for the constants A_j and B_j. Do *not* solve the equations here!

(d) Show that the BC at $y = 0$ imply that $B_j = 0$ for all $j = 0, 1, 2, \ldots$.

Hint: If

$$c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{a} = 0 \qquad \text{for all } x \in [0, a] \ , \tag{7}$$

then you can conclude that $c_j = 0$ for all $j = 0, 1, 2, \ldots$. This can be derived simply (*but you do not need to do this!*): using the fact that the system of functions $\{1, \cos\frac{\pi x}{a}, \cos\frac{2\pi x}{a}, \cos\frac{3\pi x}{a}, \ldots\}$ is orthogonal on $[0, a]$ with respect to the inner product

$$\langle f, g \rangle = \int_0^a f(x)\,g(x)\,\mathrm{d}x \ ,$$

you can multiply (7) consecutively by 1, $\cos\frac{\pi x}{a}$, $\cos\frac{2\pi x}{a}$, …, to show that all coefficients c_j must be 0.

(e) Now that you know that all $B_j = 0$ for $j = 0, 1, 2, \ldots$, impose the BC at $y = b$ to try to find the coefficients A_j. Equate the expression for $u_y(x, b)$ (with $B_j = 0$) to the function $f(x)$ from (6). What do you get for the coefficients A_n for $n \in \mathbb{N}$?

(f) The most interesting thing here is what you obtain for the coefficient A_0. Do you obtain any condition for it? What would happen if the coefficient a_0 in (6) is not equal to zero?

Hint: The answer to the last question is very dramatic!

(g) The physical reason for your dramatic answer in part (f) is that the function $f(x)$ in (5) gives the flux of heat energy through the wall at $y = b$. The coefficient a_0 of the Fourier cosine series (6) of $f(x)$ is proportional to the average of the function $f(x)$ over the interval $x \in [0, a]$: indeed,

$$\frac{1}{a}\int_0^a f(x)\,\mathrm{d}x = \frac{1}{a}\int_0^a \left(\frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{a}\right)\mathrm{d}x = \frac{a_0}{2}\ .$$

If $a_0 \neq 0$, this means that the net amount of heat going into the rectangle $[0, a] \times [0, b]$ through the wall at $y = b$ is non-zero, while the other three walls are thermally insulated. Recall that Laplace's equation describes the steady-state heat distribution. What is the physical explanation of the fact that if $a_0 \neq 0$, the BVP (5) has no solution?

Problem 3. Read Section 2.5.2 (pages 76–80) of the textbook (about the solution of the Laplace's equation in a circular disk), and solve the following BVP:

$$\begin{aligned} &\Delta u(r,\theta) = 0\ , \qquad r \in [0,2]\ , \quad \theta \in [0, 2\pi)\ , \\ &u(2,\theta) = 3 + 5\cos 6\theta - 7\sin 8\theta\ . \end{aligned} \tag{8}$$

You may simply use the general form of the solution of the problem given by Equation (2.5.45) of the book,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)\ ,$$

but please read the text from the book and identify the main points in the derivation of this expression. For example, here the discretization of the constant occurring in the separation of variables comes not from a boundary condition, but from the fact that the angular part $\Theta(\theta)$ of the solution $u(r,\theta) = R(r)\Theta(\theta)$ must be a periodic function of period 2π (these notations are different from the ones in the book). Another crucial thing to notice is that we here eliminate solutions that have a factor of r^{-n} for $n \in \mathbb{N}$ because they would lead to an unbounded growth of the solution at the origin (i.e., when $r \to 0$), which is clearly not physical.