

Problem 1. Directly from the definition, find the rates of convergence α and the asymptotic error constants λ for each of the sequences (all of which tend to 0)

$$(a) \ p_n = \frac{1}{n^2} ; \quad (b) \ p_n = 7^{-n} ; \quad (c) \ p_n = 10^{-5^n} .$$

Problem 2. The concept of rate of convergence (in particular, the order of convergence α and the asymptotic error constant λ) are very important when one is using an *iterative method*, i.e., a method in which the exact solution of the problem is found as a limit of a sequence of approximate values. If the exact value p is a limit of a sequence $\{p_n\}_{n=0}^{\infty}$ of approximate values, then the *error* at the n th step of the iteration is $E_n := |p_n - p|$. The rate of decreasing of the error is one of the most important characteristics of iterative methods of numerical computations.

Assume that the sequence $\{p_n\}_{n=0}^{\infty}$ is generated by some iterative method for finding a root of an equation. Also assume that we know that the sequence $\{p_n\}_{n=0}^{\infty}$ converges to some number p of some order α with some asymptotic error constant λ , but we don't know the values of α and λ . The goal of this problem is to develop a method for determining the numerical value of α from the numerical values of the members of the sequence $\{p_n\}_{n=0}^{\infty}$. Let $E_n := |p_n - p|$ be the error at the n th step of the iteration, and define $\ell_n := \log_{10} E_n$.

(a) Show that for large n , the following approximate identity holds:

$$\ell_n - \alpha \ell_{n-1} \approx \log_{10} \lambda .$$

Hint: Just look at the definition of order of convergence.

(b) Using the approximate identity derived in (a) show that

$$\alpha \approx \frac{\ell_n - \ell_{n+1}}{\ell_{n-1} - \ell_n} .$$

Note that this approximate formula for α does not depend on the base of the logarithms; if ℓ_n is defined as the log base 10 of E_n , the formula will remain the same.

(c) The data in the table below come from applying the so-called *Newton method* and the *secant method* to find the root of the equation

$$x + \sin x = 1 ,$$

whose exact value is $p = 0.51097342938856910952001397114508063204535889262 \dots$. Use the formula derived in part (b) to find empirically the order of convergence α for these two methods.

Table 1: Log_{10} of the errors of the Newton and the secant methods.

n	ℓ_n , Newton	ℓ_n , secant
0	-0.31067	-0.31067
1	-2.85988	-1.49389
2	-7.84087	-2.54052
3	-17.7179	-4.90935
4	-37.4715	-8.33484
5	-76.9787	-14.1282
6	-155.993	-23.3471
7	-314.022	-38.3595
8	-630.079	-62.5907
9	-1262.19	-101.834
10	-2526.42	-165.309
11	-5054.88	-268.027
12	-10111.8	-434.220

Problem 3. As discussed in class, the polynomials of order no higher than n form a linear space with respect to the addition of polynomials and multiplication of a polynomial by a number as follows: if P and Q are polynomials of degree $\leq n$ and $\alpha \in \mathbb{R}$, then the polynomials $P + Q$ and αP are defined as follows:

$$(P + Q)(x) := P(x) + Q(x) , \quad (\alpha P)(x) := \alpha P(x) .$$

Let $V_n(a, b; w(x))$ stand for the linear space of polynomials defined on the interval with left end a and right end b (at each end, the interval can be open or closed; a and b can be finite or infinite) of degree no greater than n endowed with the inner product

$$(P, Q) = \int_a^b P(x) Q(x) w(x) dx .$$

Dana defined a family of polynomials which she denoted (very modestly!) by D_0, D_1, D_2, \dots . These polynomials satisfy the following conditions:

- (i) the polynomial D_k is of degree k ;
- (ii) the coefficient of x^k in D_k is equal to 1 (such polynomials are called *monic*, following the definition on page 377 of the book);
- (iii) the polynomials $D_0, D_1, D_2, \dots, D_n$ form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = e^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^{\infty} x^k e^{-x} dx = k!$$

(where, by definition, $0! = 1$).

- (a) Clearly, $D_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial D_1 of degree 1 that is orthogonal to D_0 .
- (b) Find the only monic quadratic polynomial D_2 that is orthogonal to both D_0 and D_1 .
- (c) Show that the polynomial $P(x) = x^2 + 3$ can be represented as a linear combination of the polynomials D_0 , D_1 and D_2 as follows: $P = D_2 + 4D_1 + 5D_0$.
- (d) Show by direct integration that $(D_0, D_0) = 1$, $(D_1, D_1) = 1$, $(D_2, D_2) = 4$.
- (e) Find the orthogonal projection, $\text{proj}_{D_0+2D_1} P$, of the polynomial $P(x) = x^2 + 3$ onto the “straight line”

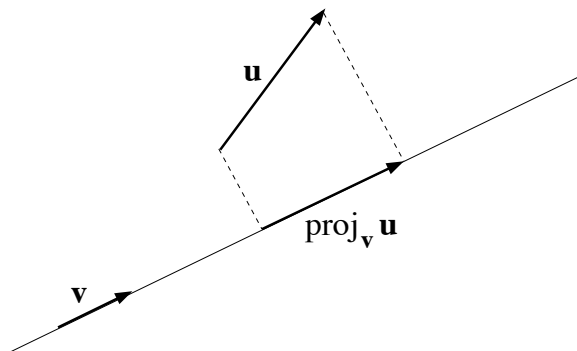
$$\ell := \{t(D_0 + 2D_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; e^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If \mathbf{u} and \mathbf{v} are vectors in the inner product linear space V , then the orthogonal projection of the vector \mathbf{u} onto the straight line in the direction of \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{(\mathbf{u}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})} \mathbf{v}$$

– see the picture below.



- (f) Finally, let $\tilde{D}_k := \mu_k D_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\|\tilde{D}_k\| := \sqrt{(\tilde{D}_k, \tilde{D}_k)},$$

of the polynomial \tilde{D}_k is 1. Find the explicit expressions for $\tilde{D}_0(x)$, $\tilde{D}_1(x)$, and $\tilde{D}_2(x)$.

Problem 4. Hilbert matrices are a family of matrices that have very high condition numbers. The Hilbert matrix $H^{(n)}$ of size $n \times n$ has matrix elements

$$h_{ij}^{(n)} = \frac{1}{i+j-1}.$$

The MATLAB command to generate a Hilbert matrix of size \mathbf{n} is `hilb(n)`. In this problem you will use MATLAB to study the reliability of the residual and the relative residual as a predictor of the error of an approximate solution of the linear system $H^{(11)}\mathbf{x} = \mathbf{b}$.

- (a) In MATLAB, let \mathbf{A} denote the Hilbert matrix $H^{(11)}$, and `xexact` stand for the exact solution $\mathbf{x}_{\text{exact}} = (1 \ 2 \ 3 \ \dots \ 10 \ 11)^T$. Define the right-hand side \mathbf{b} by typing `b = A*xexact`, and then let `xapprox` be the solution of the linear system $H^{(11)}\mathbf{x} = \mathbf{b}$ found by MATLAB (you can find `xapprox` by using the MATLAB command `inv` to invert the matrix \mathbf{A} , no need to write your own program).
- (b) Find the error $\mathbf{e} = \mathbf{x}_{\text{approx}} - \mathbf{x}_{\text{exact}}$ and the residual $\mathbf{r} = H^{(11)}\mathbf{x}_{\text{approx}} - \mathbf{b}$, and the ℓ_∞ -norms of \mathbf{b} , $\mathbf{x}_{\text{exact}}$, \mathbf{e} , and \mathbf{r} , as well as the relative error and the relative residual.
- (c) Use the MATLAB command `norm` to find $\|H^{(11)}\|_\infty$, $\|(H^{(11)})^{-1}\|_\infty$, and $\kappa_\infty(H^{(11)})$.
- (d) Check that all inequalities proved in the Theorem on page 182 of the book hold in the particular example in this problem. Please write a clear and detailed solution.