MATH 4073   Homework 4   Due $\boxed{\text{in class}}$ on Fri, 09/30/16

Problem 1. [Rate of convergence of a sequence, Aitken's Δ^2 method]

Let the sequence $\{p_n\}_{n=0}^{\infty}$ be defined for all $n \geq 0$ by

$$p_0 = 5 , \qquad p_n = \frac{7p_{n-1}}{8} + \frac{1}{p_{n-1}^2} \quad \text{for } n \in \mathbb{N} .$$

(a) Think of the sequence $\{p_n\}_{n=0}^{\infty}$ as defined by a functional iteration, $p_n = g(p_{n-1})$, for an appropriate function g. Write g explicitly. Use some of the theorems about Fixed Point Iteration to show that g has a fixed point in the interval $[\frac{3}{2}, 100]$. (In fact, instead of 100, I could have put any arbitrarily large positive number.) Please specify which theorem you have used.

(b) Use some of the theorems about Fixed Point Iteration to show that there is only one fixed point of g in the interval $[\frac{3}{2}, 100]$. Again, do not forget to specify which theorem you have used.

(c) Solve the equation $x = g(x)$ by hand to find the fixed points of g. Observe that the value that you just found, as well as the value of p_0 are in the interval $[\frac{3}{2}, 100]$. This implies that there are no fixed points of g in this interval other than the value p that you have found. This, in turn, implies the that the sequence $\{p_n\}_{n=0}^{\infty}$ will converge to the fixed point p.

(d) What does the general theory predict about the rate of convergence α and the asymptotic error constant λ?

(e) I have run the MATLAB code `fixedpoint.m` (taken from the class web-site) for the functional iteration $p_n = g(p_{n-1})$. with g being the function you wrote in part (a), and with initial value $p_0 = 5$. Part of the output is shown on the next page.

Use the method from Problem 3 of Homework 3 to compute empirically the values of the rate of convergence α and the asymptotic error constant λ. In your calculation, use values of p_n with n around 30. Please write your computations explicitly, with the concrete values you are using. Do the empirical values match the predictions of the general theory from part (d)?

(f) Clearly, the fixed-point iteration above converges, but the convergence is not very fast. Use Aitken's Δ^2 method for accelerating convergence,

$$\widehat{p}_n := \{\Delta^2\}(p_n, p_{n+1}, p_{n+2}) := p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} := p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} + p_n - 2p_{n+1}}$$

to compute the value of $\widehat{p}_{15}$. Find the errors $|\widehat{p}_{15} - p|$ and $|p_{15} - p|$; discuss briefly what you observe.

Remark: When computing the value of $\widehat{p}_{15}$, use more digits, otherwise you will lose precision because of round-off error.

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```
>> fixedpoint(inline('7/8*x+1/x^2'), 5.0, 1e-8, 1000, 1)

iters        x                 x-xold       log_10(|x-xold|)

   1     4.41500000000000    -0.58500000000000     -0.23284
   2     3.91442750651221    -0.50057249348779     -0.30053
   3     3.49038653584740    -0.42404097066481     -0.37259
   4     3.13617116779797    -0.35421536804943     -0.45073
   5     2.84582156525324    -0.29034960254473     -0.53708
   6     2.61357047114044    -0.23225109411279     -0.63404
   7     2.43327095777499    -0.18029951336545     -0.74401
   8     2.29800796769249    -0.13526299008251     -0.86882
   9     2.20012076284642    -0.09788720484607     -1.00927
  10     2.13169455687759    -0.06842620596883     -1.16478
  11     2.08529726669286    -0.04639729018473     -1.33351
  12     2.05460133119119    -0.03069593550168     -1.51292
  13     2.03466515118882    -0.01993618000236     -1.70036
  14     2.02188593661876    -0.01277921457006     -1.89350
  15     2.01376722909088    -0.00811870752788     -2.09051
  16     2.00863973290306    -0.00512749618782     -2.29009
  17     2.00541374881800    -0.00322598408506     -2.49134
  18     2.00338906862121    -0.00202468019678     -2.69364
  19     2.00212031661768    -0.00126875200354     -2.89662
  20     2.00132603964781    -0.00079427696987     -3.10003
  21     2.00082910418512    -0.00049693546268     -3.30370
  22     2.00051831893458    -0.00031078525055     -3.50754
  23     2.00032399968943    -0.00019431924514     -3.71148
  24     2.00020251948461    -0.00012148020483     -3.91549
  25     2.00012658236699    -0.00007593711761     -4.11955
  26     2.00007911698345    -0.00004746538354     -4.32362
  27     2.00004944928825    -0.00002966769520     -4.52772
  28     2.00003090626362    -0.00001854302463     -4.73182
  29     2.00001931659386    -0.00001158966976     -4.93593
  30     2.00001207294112    -0.00000724365274     -5.14004
  31     2.00000754561553    -0.00000452732559     -5.34416
  32     2.00000471602038    -0.00000282959515     -5.54828
  33     2.00000294751691    -0.00000176850347     -5.75239
  34     2.00000184219970    -0.00000110531721     -5.95651
  35     2.00000115137545    -0.00000069082425     -6.16063
  36     2.00000071960990    -0.00000043176554     -6.36475
  37     2.00000044975629    -0.00000026985362     -6.56887
  38     2.00000028109772    -0.00000016865857     -6.77299
  39     2.00000017568609    -0.00000010541163     -6.97711
  40     2.00000010980381    -0.00000006588228     -7.18123
  41     2.00000006862738    -0.00000004117643     -7.38535
```

Problem 2. [**Newton's method for multiple zeros**]

Recall that the *multiplicity* of a zero p of the function f is defined as the number m such that

$$f(x) = (x-p)^m q(x) \; ,$$

where q is a function satisfying $\lim_{x \to p} q(x) \neq 0$.

Recall also that Newton's method for finding a zero of the function f (or, equivalently, a root of the equation $f(x) = 0$) is based on the iterative procedure $p_n = g(p_{n-1})$, where p_0 is some starting value, and $g(x) = x - \dfrac{f(x)}{f'(x)}$. We stated in class that, if p is a simple zero of f (i.e., a zero of multiplicity 1) and the Newton's method converges to p, then the convergence is at least quadratic, i.e., or order $\alpha \geq 2$.

If, however, the zero of f is non-simple, then the Newton's method converges only linearly.

In class we proved that, if p is a fixed point of the function g and $g'(p) \neq 0$, then if the iteration $p_n = g(p_{n-1})$ converges to p, then the convergence is linear and the asymptotic error constant is $\lambda = |g'(p)|$.

In this problem you will show that, indeed, the Newton's method converges linearly for $m \geq 2$, and will find a modification of Newton's method that works with multiple zeros (but one needs to know the multiplicity of the zero and pass it to the program as one of the arguments).

Let p be a zero of multiplicity $m \geq 2$ of f. Then the Newton's iteration for finding a zero of f has the form

$$
\begin{aligned}
g(x) &= x - \frac{f(x)}{f'(x)} \\[2mm]
&= x - \frac{(x-p)^m q(x)}{[(x-p)^m q(x)]'} \\[2mm]
&= x - \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)} \\[2mm]
&= x - (x-p) \frac{q(x)}{mq(x) + (x-p)q'(x)} \; ,
\end{aligned}
$$

therefore

$$
g'(x) = 1 - \frac{q(x)}{mq(x) + (x-p)q'(x)} - (x-p) \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{q(x)}{mq(x) + (x-p)q'(x)} \right) \; .
$$

This implies that

$$
g'(p) = 1 - \frac{1}{m} \neq 0 \; ,
$$

hence the convergence of Newton's method is only linear.

(a) Let p be a zero of multiplicity $m \geq 2$ of f. Consider the following modification of the Newton's method: $p_n = g(p_{n-1})$, where

$$
g(x) = x - m \frac{f(x)}{f'(x)} \; .
$$

Show that in this case $g'(p) = 0$, hence the convergence is faster than linear.

(b) Show that the multiplicity of the root $\frac{\pi}{2}$ of the equation $(x - \frac{\pi}{2})(1 - \sin x) = 0$ is $m = 3$.

   Hint: Expand $\sin x$ in a Taylor series around $p_0 = \frac{\pi}{2}$.

(c) The Mathematica code

```
p = N[3, 50000];
m = 3;
f[x_] := (x - Pi/2) * (1-Sin[x]);
For[i = 1, i <= 10, i++,
  { p = p - m*f[p]/f'[p],
     error = Abs[p - Pi/2],
     Print[i, "   ", N[Log[error],10]]
  }
]
```

can be used to find empirically the order of convergence of the method; note that it does the calculations with accuracy or 50000 decimal digits! Run this code, <u>attach the printout</u>. Use the data from the printout to compute the order of convergence. Explain how you found the order of convergence, and write explicitly all calculations.

(d) The number $\frac{\pi}{2}$ is a root of the equation

$$\left(x - \frac{\pi}{2}\right)^3 (1 - \sin x) = 0$$

of multiplicity 5. Modify the Mathematica code from part (c) to find empirically the order of convergence of the modified Newton's method for this equation. Run your code, <u>attach the printout</u>, and use the data from the printout to compute the order of convergence.

Problem 3. [Synthetic division and deflation for computing roots of polynomials]

In this problem you will use the Horner's method, called also *synthetic division*, to find all the zeros of a polynomial of degree 5, i.e., all the roots of the equation

$$x^5 - 8x^4 + 36x^3 - 42x^2 - 37x + 50 = 0 .$$ (1)

(a) Use synthetic division to find

$$\frac{x^5 - 8x^4 + 36x^3 - 42x^2 - 37x + 50}{x + 1} .$$

(b) Use synthetic division to find

$$\frac{x^5 - 8x^4 + 36x^3 - 42x^2 - 37x + 50}{(x + 1)(x - 2)} .$$

(c) Use Horner's method to find the value of the cubic polynomial found in part (b) at $x_0 = 1$. This should suggest a way of representing the cubic polynomial from part (b) as a product of a linear polynomial and a quadratic polynomial.

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(d) Use the results of parts (a), (b), and (c) to find all the roots of the equation (1) (including the non-real roots).

Remark: The built-in MATLAB command `roots` finds (possibly complex) roots of polynomials. The argument of `roots` is a vector of all coefficients of the polynomial, starting with the one at the highest power. For example, to find all roots of the polynomial $2x^5 + x^4 + 5x^2 - 13x + 5$, type `roots([2 1 0 5 -13 5])` and press RETURN.

One can find out more information about MATLAB functions, say `roots`, by typing

    help roots

Use this to see the description of MATLAB command `fzero` which is a built-in zero finder. By typing

    help help

you will get information about the command `help` itself, and

    help /

will give you a description of all operators and special characters.