

Problem 1. Directly from the definition, find the rates of convergence α and the asymptotic error constants λ for each of the sequences (all of which tend to 0)

(a) $x_n = \frac{1}{n^2}$; (b) $x_n = 7^{-n}$; (c) $x_n = 10^{-5^n}$.

Problem 2. In this problem we will find the value of the number $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}}}}$, and will study how a certain sequence converges to it.

- (a) Show that the numbers 2 and 4 satisfy the equation $\sqrt{2}^x = x$.
- (b) Consider the function $g(x) := \sqrt{2}^x$, $x \geq 0$. Use derivatives to show that g is an increasing concave up function for $x \in [0, \infty)$. (To differentiate g , note that $\sqrt{2}^x = e^{\frac{\ln 2}{2} x}$.)

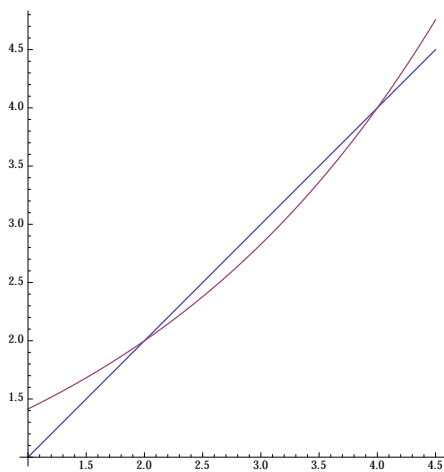


Figure 1: Plots of the diagonal and the graph of $g(x) = \sqrt{2}^x$.

- (c) Use what you found in (b) to convince me that the equation $\sqrt{2}^x = x$ has no other solutions in $[0, \infty)$ except 2 and 4.
- (d) Is the fixed point $x = 2$ of the function attracting or repelling? How about $x = 4$? Justify your claims (a pictorial “proof” is enough).
- (e) Let $x_0 := 1$, $x_n := \sqrt{2}^{x_{n-1}}$ for $n \geq 1$. Perform several iterations by using Mathematica, MATLAB, or any other software. Does the sequence $(x_n)_{n=0}^\infty$ seem to converge? To what value? (There is no need to attach a printout, just tell me what you observe.)
- (f) Let x_* be the limit of the sequence defined in (e). Show theoretically that the limit

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|}$$

exists. What does the theory predict about the value of this limit?

Problem 3. In this problem you will construct a piecewise-linear and a quadratic Lagrange interpolating polynomials for the function $\cos(\pi x)$ on the interval $[0, \frac{1}{2}]$.

Let $f(x) = \cos(\pi x)$, and $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$. You will need to use the values of $f(x)$ at these points: $y_0 = \cos(0) = 1$, $y_1 = \cos \frac{\pi}{3} = \frac{1}{2}$ and $y_2 = \cos \frac{\pi}{2} = 0$. Figure 2 shows the graphs of $f(x)$ and the interpolating polynomials you will compute.

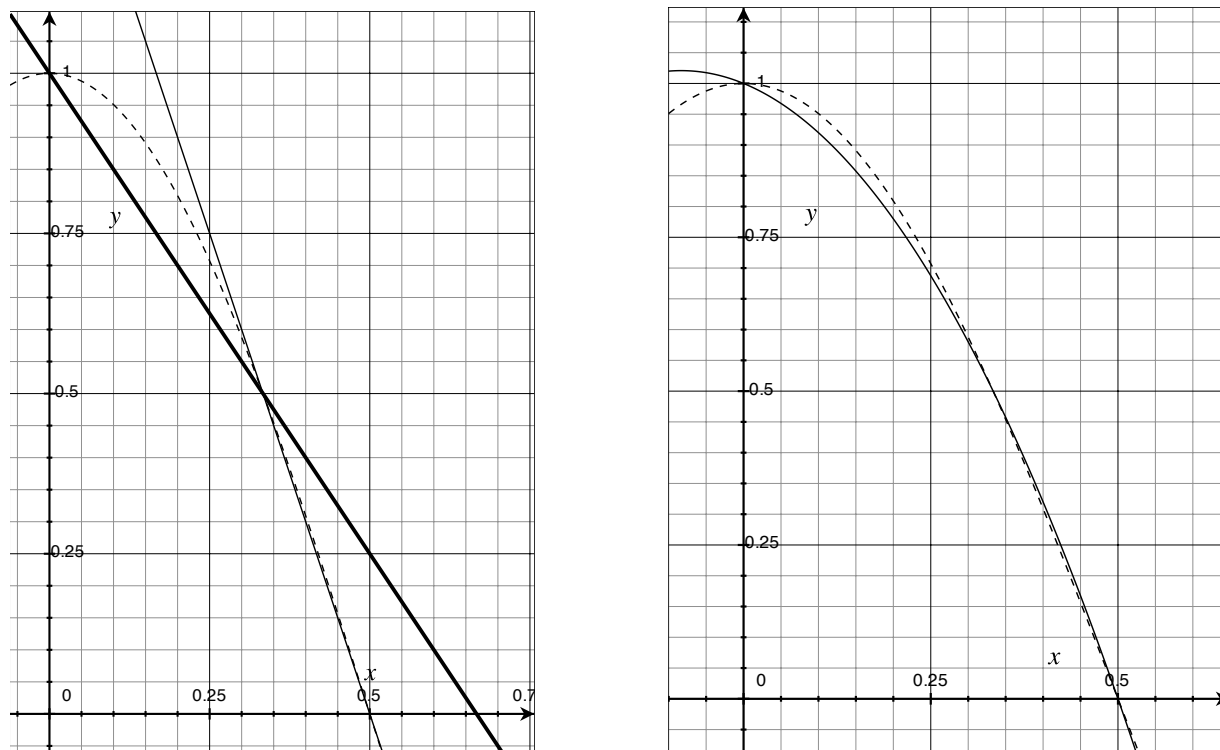


Figure 2: (LEFT) Piecewise-linear interpolation of $\cos \pi x$ (the function $\cos \pi x$ is plotted with dashed line, the interpolating linear function on $[0, \frac{1}{3}]$ with a thick solid line, and the interpolating linear function on $[\frac{1}{3}, \frac{1}{2}]$ with a thin solid line). (RIGHT) Quadratic interpolation of $\cos \pi x$ on $[0, \frac{1}{2}]$ using the points $(0, 1)$, $(\frac{1}{3}, \frac{1}{2})$, and $(\frac{1}{2}, 0)$.

(A) Piecewise-linear interpolation.

- (A1) Find the piecewise-linear interpolation of the function $f(x)$ on the interval $[0, \frac{1}{2}]$ based on the values of the function at the points 0 , $\frac{1}{3}$, and $\frac{1}{2}$. Naturally, you will have to find two different first-order Lagrange polynomials – one in the interval $[0, \frac{1}{3}]$, and another one on $[\frac{1}{3}, \frac{1}{2}]$:

$$P_{\text{piece-lin}}(x) = \begin{cases} b_1x + c_1, & x \in [0, \frac{1}{3}] , \\ b_2x + c_2, & x \in [\frac{1}{3}, \frac{1}{2}] . \end{cases}$$

- (A2) Use the piecewise-linear interpolating polynomial found in part (A1) to compute the approximate value of $\cos \frac{\pi}{6}$, and use it to find the numerical value of the absolute error $|\cos \frac{\pi}{6} - P_{\text{piece-lin}}(\frac{1}{6})|$.

(B) Quadratic Lagrange interpolation.

- (B1) Construct the Lagrange interpolating polynomial of degree at most 2, $P_2(x)$, to $\cos(\pi x)$ in the interval $[0, \frac{1}{2}]$. The polynomial $P_2(x)$ is the only quadratic function whose graph goes through the points $(0, \cos 0) = (0, 1)$, $(\frac{1}{3}, \cos \frac{\pi}{3}) = (\frac{1}{3}, \frac{1}{2})$ and $(\frac{1}{2}, \cos \frac{\pi}{2}) = (\frac{1}{2}, 0)$.
- (B2) Use the quadratic Lagrange interpolating polynomial found in part (B1) to compute the approximate value of $\cos \frac{\pi}{6}$. Find the numerical value of the absolute error $|\cos \frac{\pi}{6} - P_2(\frac{1}{6})|$.

Problem 4. This problem is a continuation of Problem 3. Let

$$I_{\text{exact}} := \int_0^{\frac{1}{2}} \cos(\pi x) dx ,$$
$$I_{\text{piece-lin}} := \int_0^{\frac{1}{2}} P_{\text{piece-lin}}(x) dx , \quad I_{\text{quadr}} := \int_0^{\frac{1}{2}} P_2(x) dx$$

be the definite integrals from 0 to $\frac{1}{2}$ of the function $\cos(\pi x)$ and the piecewise-linear and the quadratic interpolating functions you computed in Problem 3.

- (a) Without computing anything, decide which of the numbers I_{exact} and $I_{\text{piece-lin}}$ is larger.

Why can't you use the similar simple reasoning to decide which of the numbers I_{exact} and I_{quadr} is larger?

Hint: Look at Figure 2 and think about the concavity of $\cos(\pi x)$ for $x \in [0, \frac{1}{2}]$.

- (b) Compute the numerical values of I_{exact} , $I_{\text{piece-lin}}$, and I_{quadr} . Was your prediction in part (a) correct?

Hint: Be careful when you compute $I_{\text{piece-lin}}$; the value of this integral is $\frac{7}{24} \approx 0.291667$.

- (c) Compute the numerical values of the absolute errors in approximating I_{exact} by $I_{\text{piece-lin}}$ and by I_{quadr} .

Problem 5. The purpose of this problem is to construct and study the Newton's divided difference form of the interpolating polynomial,

$$P_n(x) = f[x_0] + \sum_{k=1}^n f[x_0, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j) ,$$

to the function $f(x) = \cos(\pi x)$. The points x_i , $i = 0, 1, 2, 3$ used to construct the interpolating polynomial are given in the table below (x_0 , x_1 and x_2 are the same as in the previous problems). The graphs of the function $f(x)$ and the interpolating polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$ are plotted in Figure 3 below.

- (a) Compute the missing entries in the divided differences table below. Write your calculations clearly and leave the coefficients in symbolic form (i.e., do not compute the *numerical values* of things like $12(8\sqrt{2} - 11)$).

x_i	0 th order	1 st order	2 nd order	3 rd order
$x_0 = 0$	$f[x_0] = 1$			
		$f[x_0, x_1] = ?$		
$x_1 = \frac{1}{3}$	$f[x_1] = \frac{1}{2}$		$f[x_0, x_1, x_2] = ?$	
		$f[x_1, x_2] = ?$		$f[x_0, x_1, x_2, x_3] = 12(8\sqrt{2} - 11)$
$x_2 = \frac{1}{2}$	$f[x_2] = 0$		$f[x_1, x_2, x_3] = 12(2\sqrt{2} - 3)$	
		$f[x_2, x_3] = -2\sqrt{2}$		
$x_3 = \frac{1}{4}$	$f[x_3] = ?$			

- (b) Write down the interpolating polynomial $P_0(x)$ based on the values in the divided differences table above.
- (c) Similarly to part (b), write down the interpolating polynomial $P_1(x)$ based on the values in the divided differences table above.
- (d) Similarly to part (b), write down the interpolating polynomial $P_2(x)$ based on the values in the divided differences table above. Do *not* expand it – just substitute the coefficients in the Newton’s divided difference interpolating polynomial with the corresponding entries from the table.
- (e) Similarly to part (b), write down the interpolating polynomial $P_3(x)$ based on the values in the divided differences table above. Do *not* expand the polynomial!

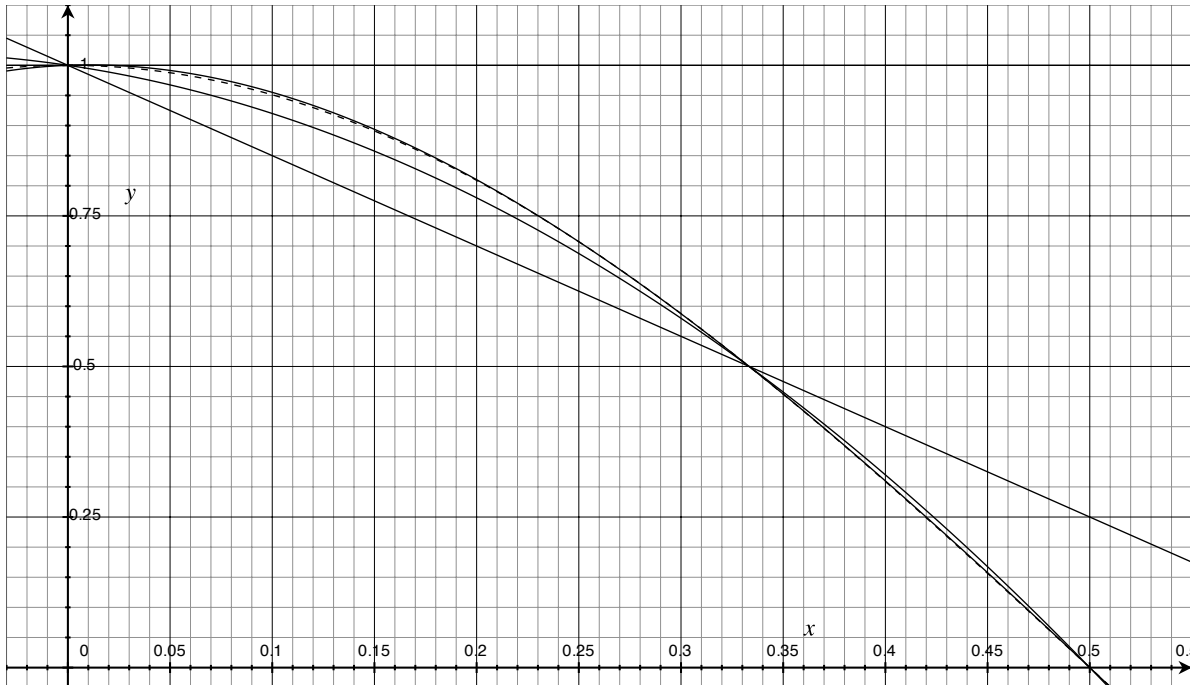


Figure 3: Graphs of the function $f(x)$ (the dashed line) and the interpolating polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$.