MATH 3423 Homework 4 Due Thu, 09/19/19

Problem 1. Find all 4th roots of i.

Problem 2. Directly from the definition of $\sinh z$, show that the Taylor expansion of $\sinh z$ around 0 is

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \ .$$

Problem 3. Determine $\ln(-\mathrm{i})$ and $\mathrm{Ln}\,(-\mathrm{i})$.

Problem 4.

(a) Show that $\arcsin z = -\mathrm{i}\,\ln\left(\mathrm{i}z \pm \sqrt{1-z^2}\right)$.

Hint: Solve $\sin w = z$ (i.e., $\frac{\mathrm{e}^{\mathrm{i}w}-\mathrm{e}^{-\mathrm{i}w}}{2\mathrm{i}} = z$) for z. You can set $\xi := \mathrm{e}^{\mathrm{i}w}$, and rewrite $\frac{\mathrm{e}^{\mathrm{i}w}-\mathrm{e}^{-\mathrm{i}w}}{2\mathrm{i}} = z$ as a quadratic equation for ξ.

(b) Use your result from part (a) to solve the equation $\sin w = 2$. Note that this equation has no solution if w is real.

(c) Directly from the definition of the sine function, show that $\sin\left(\frac{\pi}{2} - \mathrm{i}\,\ln\left(2 \pm \sqrt{3}\right)\right) = 2$.

Problem 5. Evaluate $\sqrt{-1}^{\sqrt{-1}}$ (i.e., i^{i}).

Problem 6. Show that $\lim\limits_{z\to 0} \dfrac{z^*}{z}$ does not exist by taking the limit along the ray $y = \alpha x$, where α is a real constant.

Problem 7. Find all singularities of $f(z) = \dfrac{z}{z^4+16}$ and plot them in the complex plane.

Problem 8. Let $u(x,y) = x^3 - 3xy^2$. In this problem you will use the Cauchy-Riemann equations to find a function $v(x,y)$ such that the function $f(z) = u(x,y) + \mathrm{i}\,v(x,y)$ is analytic (where $z = x + \mathrm{i}y$). Please follow the steps below.

(a) Consider the first Cauchy-Riemann equation,

$$u_x(x,y) = v_y(x,y) \ . \tag{1}$$

You know the left-hand side, so that you can consider it as a differential equation for $v(x,y)$. Integrate both sides with respect to y:

$$v(x,y) = \int u_x(x,y)\,\mathrm{d}y \tag{2}$$

to obtain an expression for $v(x, y)$. Since the integration in the right-hand side is over the variable y, treat the variable x as a parameter independent of y (i.e., when integrating, treat x as a constant). When you compute an indefinite integral of the form $\int f(y)\,\mathrm{d}y$, the result contains one arbitrary constant. In computing the integral in the right-hand side of (2), since there is another variable (namely, x), instead of an arbitrary constant, you will have to write an arbitrary function of x, say, $\varphi(x)$. The function $\varphi(x)$ is arbitrary for now – it will be determined in part (b).

(b) Now consider the second Cauchy-Riemann equation,

$$u_y(x, y) = -v_x(x, y) \ . \tag{3}$$

You already have an expression for $v(x, y)$ (obtained in part (a)), in which the only thing that you still do not know is the function $\phi(x)$. Use (3) to find the function $\phi(x)$. The function $\phi(x)$ that you will find will contain one arbitrary constant – this time this constant will be a true constant (independent of both x and y) – this is to be expected (because the Cauchy-Riemann equations (1) and (3) do not tell us anything about the *function* $v(x, y)$, but only about its partial derivatives).

(c) Write down the expression $u(x, y) + \mathrm{i}v(x, y)$ with the concrete functions $u(x, y)$ and $v(x, y)$. For these particular functions $u(x, y)$ and $v(x, y)$, write down

$$f(z) = u(x, y) + \mathrm{i}v(x, y)$$

only in terms of $z = x + \mathrm{i}y$, but not as a function of x and y separately. (Hint: The function $f(z)$ is quite simple.)

(d) In the rest of this problem you will obtain the same function $v(x, y)$ that you found in part (b), but performing the steps in a different order. First consider the second Cauchy-Riemann equation (3), and integrate both sides with respect to x to obtain

$$v(x, y) = -\int u_y(x, y)\,\mathrm{d}x \ .$$

This time the expression for $v(x, y)$ you obtain will contain an arbitrary function of y, let's call it $\psi(y)$, which you will find in the next part of the problem.

(e) Now plug the function $v(x, y)$ obtained in part (d) into the first Cauchy-Riemann equation (1), in order to derive a differential equation for $\psi(y)$. Solve the equation to find $\psi(y)$ and, hence, $v(x, y)$.

Problem 9. Can the function $u(x, y) = x^4 + 3x^2y^2$ be the real part of an analytic function $f(z)$? Support your answer with and explanation and all the necessary calculations.

Problem 10. In evaluating integrals along contours in the complex plane, we will need to use some of the techniques from the calculus of real variables, so this problem will refresh your memory. Let $\mathbf{F}(x, y) = (2x^2 + y)\,\mathbf{i} - 5xy\,\mathbf{j}$ be a vector field in $\mathbb{R}^2$. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ from $(0, 0)$ to $(2, 4)$ along the path C given by the parametric equations $\mathbf{r}(t) = (x(t), y(t)) = (t, t^2)$.

Hint: Recall that, by definition, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t$, where t_i and t_f are the initial and final values of the parameter t.