MATH 3423 Homework 4 Due Fri, 9/27/13

Before you start working on this homework, please read the Food for Thought Problem at the end of this homework.

Problem 1. Let C be the circle of radius R centered at the origin, traversed in positive (i.e., counterclockwise) direction.

(a) Compute the value of integral $\oint_C z^* \, \mathrm{d}z$.

Hint: The most efficient way of parameterizing C is $z(\theta) = R\mathrm{e}^{\mathrm{i}\theta}$, where the parameter θ varies from 0 to 2π.

(b) Based solely on your result from part (a), can you say for sure whether the function z^* is analytic in the domain surrounded by C? Explain briefly how you came to your conclusion.

Problem 2. Use one of the results given on page 883 to give an upper bound for $\left|\int_C \frac{\mathrm{d}z}{z^3}\right|$, where C is the straight line segment from i to $3+5i$.

Problem 3.

(a) Directly from the definition, calculate the integral $\int_C z \, \mathrm{d}z$, where the path C is given by the parametric equations $z(t) = t + \mathrm{i}\,\mathrm{e}^t$, and the parameter t varies from 0 to 1.

(b) A much faster way to compute the integral from part (a) is to use the Theorem on page 888 of the book. Compute the value of the integral, and compare your result with the result from part (a). Did you use the specific form of the contour C in your calculation in this part of the problem?

(c) Evaluate the value of the integral $\int_C \left(z^6 + z\mathrm{e}^{z^2}\right) \mathrm{d}z$, where the path C is a segment of a straight line from starting at 2 and ending at i.

Problem 4. Evaluate $\oint_C \frac{\mathrm{d}z}{(z-a)^5}$, where C is a simple closed contour (of arbitrary shape) encircling the point a.

Problem 5. Evaluate $\oint_C \frac{z\,\mathrm{e}^{-z}}{z^3+8}\,\mathrm{d}z$, where C is the unit circle centered at the origin and traversed in positive direction.
Hint: Before you roll up your sleeves and start calculating, think! Or maybe think *instead of* rolling up your sleeves...

Problem 6. Use Cauchy's Integral Formula to evaluate $\oint_C \frac{\tan z}{4z-\pi}\,\mathrm{d}z$, where C is the contour (traversed in positive direction) given by

(a) $|z| = \frac{1}{2}$;

(b) $|z| = 1$.

Problem 7. Evaluate $\oint_C \frac{\mathrm{e}^{iz}}{z^2+1}\,\mathrm{d}z$, where C is the contour given by $|z| = 2$ (traversed in positive direction).

Problem 8. Evaluate $\oint_C \frac{\cos z}{z(z-2i)^2}\,\mathrm{d}z$, where C is the circle $|z-3i| = 2$ (traversed in positive direction).
Hint: See Example 4 on page 897. Be careful about which singularities lie inside C.

Problem 9.

(a) For which values of z does the series $\sum_{n=0}^{\infty} \mathrm{e}^{nz}$ converge?

(b) What is the sum of the series from (a)? (Of course, the sum will be a function of z.)

Problem 10. Determine the disk of convergence of the power series $\sum_{n=0}^{\infty} \frac{(z-i)^n}{n^2}$.

Problem 11.

(a) Find the Taylor expansion of the function $f(z) = \frac{1}{1+z^2}$ about $z = 0$ by using the formula for the sum of a geometric series applied to $\frac{1}{1+z^2} = \frac{1}{1-(-z^2)}$.

(b) Use your result from (a) and the fact that $\arctan\zeta = \int_0^{\zeta} \frac{\mathrm{d}z}{1+z^2}$ to obtain the Taylor expansion of $\arctan\zeta$ about $\zeta = 0$.

Food for Thought Problem – NOT TO BE TURNED IN!

Let $\mathbf{F}(x, y) = (2x^2 + y)\,\mathbf{i} - 5xy\,\mathbf{j}$ be a vector field in $\mathbb{R}^2$. Evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ from $(0, 0)$ to $(2, 4)$ along the path C given by the parametric equations $\mathbf{r}(t) = (x(t), y(t)) = (t, t^2)$.

Hint: Recall that, by definition, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t$, where t_i and t_f are the initial and final values of the parameter t. We have $t_\mathrm{i} = 0$, $t_\mathrm{f} = 2$, $\mathbf{r}'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(t, t^2) = \mathbf{i} + 2t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 3t^2\,\mathbf{i} - 5t^3\,\mathbf{j}$, therefore

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^2 \left(3t^2\,\mathbf{i} - 5t^3\,\mathbf{j}\right) \cdot (\mathbf{i} + 2t\,\mathbf{j})\ \mathrm{d}t \\ &= \int_0^2 \left(3t^2 - 10t^4\right)\ \mathrm{d}t = \left(t^3 - 2t^5\right)\Big|_{t=0}^{2} = 8 - 2 \cdot 32 = -56\ .\end{aligned}$$