Additional Problem.

Let $\mathcal{L}$ stand for the set of all linear functions of one variable, i.e., $\mathcal{L}$ consists of all functions of the form $f(x) = ax + b$ (where a and b are constants). Clearly, the quadratic function $q(x) = x^2$ does not belong to $\mathcal{L}$. Suppose that you would like to find a linear function f from $\mathcal{L}$ that is the best approximation of q on the interval $[0, 1]$, in the sense that f is such that the "error" defined by the integral

$$\int_0^1 [q(x) - f(x)]^2 \, \mathrm{d}x = \int_0^1 [q(x) - (ax + b)]^2 \, \mathrm{d}x = \frac{a^2}{3} + b^2 + ab - \frac{a}{2} - \frac{2b}{3} + \frac{1}{5} \, .$$

(a) Find the values of a and b for which the "error" is the smallest.

Hint: The answer is $a = 1$, $b = -\frac{1}{6}$, but you have to show me how you can obtain these values.

(b) Based on your answer in part (a) (which was actually given in the hint), write down the function f from $\mathcal{L}$ that is "closest" to the function q, where by "distance" between f and q we mean the "error" defined in part (a).

(c) What is the value of the "error" in approximating the function $q(x)$ with the function obtained in part (b)?