MATH 5763 Homework 3 Due Thu, 2/7/2019

Problem 1. A pair of dice is rolled until a sum or either 5 or 7 appears. In this problem we will find the probability that a sum equal to 5 occurs first. Please follow the steps below.

(a) What is the probability that *in one individual roll of the two dice* the sum will be 5? Four your convenience, the table below represents all possible outcomes. Think of the two dice as being distinct (say, one of them is red and the other is green). Then the first number in each pair represents the outcome of the red die, and the second one represents the outcome of the green die.

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

(b) What is the probability that *in one individual roll of the two dice* the sum will be neither 5 nor 7?

(c) Let E_n be the event that in the sequence of rolls a 5 occurs *for the first time* on the nth roll, and no 7 has occurred before that. (In other words, E_n is the event that a 5 occurs on the nth roll and no 5 or 7 occurs in the first $n-1$ rolls.) Find the probability $\mathbb{P}(E_n)$ of the event E_n.

(d) Argue that the desired probability (i.e., the probability that a 5 occurs first) is equal to the infinite sum $\displaystyle\sum_{n=1}^{\infty} \mathbb{P}(E_n)$.

(e) Find the value of the desired probability by calculating the above sum. You may need the formula for the sum of a geometric series, $\displaystyle\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, valid whenever $|q| < 1$.

Problem 2. A die is rolled repeatedly. Explain in a couple of sentences why the following are Markov chains, and find their 1-step transition probability matrices $\mathbf{P}$.

(a) The number X_n of sixes in the first n rolls.

(b) At time n, the time X_n since the most recent six.

Hint: The state space for both Markov chains is the set of non-negative integers, $\mathbb{Z}_+ := \{0, 1, 2, 3, \ldots\}$ (but the 1-step transition probability matrices are different).

Problem 3. Let X be a stationary discrete-time discrete-state space Markov chain with state space S consisting of two states, 0 and 1, and let the 1-step transition probability matrix of the stochastic process be

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix}, \tag{1}$$

where $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$. Assume that you do not know the exact value of the initial value X_0 of the MC X, but you know that

$$\mathbb{P}(X_0 = 0) = \frac{1}{5}. \tag{2}$$

(a) Find the p.m.f. p_{X_0} of the initial state X_0 of the MC.

(b) Find $\mathbb{E}[X_0]$ and $\operatorname{Var} X_0$.

(c) Find the p.m.f. p_{X_1} of the state X_1 of the MC at time 1.

(d) Find $\mathbb{E}[X_1]$.

(e) Find the p.m.f. p_{X_2} of the state X_2 of the MC at time 2.

Problem 4. Consider a Markov chain whose state space consists of five states: α, β, γ, δ, ϵ, and whose 1-step transition probability matrix is the following:

$$\mathbf{P} = \begin{array}{c} \begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \epsilon \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{array}$$

(a) Draw a diagram with arrows (where each arrow from state i to state j represents a nonzero probability p_{ij}), and identify the transient and the recurrent states (do not do any computations yet). You will find that two states are transient (denote the set of transient states by D), and there will be two closed and irreducible sets of recurrent states (one of them – call it C_1 – will consist of two states, and the other will consist of only one state – call this set C_2).

(b) Now relabel the states α, β, γ, δ, ϵ as 1, 2, 3, 4, 5, in such a way that the $C_1 = \{1, 2\}$, $C_2 = \{3\}$, and the states 4 and 5 to be the transient states, i.e., $D = \{4, 5\}$. In C_1, let state 1 be the state with one-step probability for transition to itself equal to $\frac{1}{3}$; in D, let state 4 be the state with nonzero one-step probability for transition to itself.

(c) Carefully write down all entries in the one-step transition probability matrix $\widetilde{\mathbf{P}}$ with the relabeled states. It should look like this:

$$\widetilde{\mathbf{P}} = \left(\begin{array}{c|c|c} \mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}_2 & \mathbf{0} \\ \hline * & * & \mathbf{T} \end{array} \right),$$

where $\mathbf{0}$ are matrices (of appropriate size) with all entries equal to zero, while the stars represent matrices that are generally not zero (but nothing more concrete can be said about them in general).

Check that $\mathbf{C}_1$ and $\mathbf{C}_2$ are stochastic matrices, while $\mathbf{T}$ is *not* a stochastic matrix.

Hint: The matrix $\mathbf{C}_1$ is the same as the matrix of the 2-state Markov chain from Example 3.2.2 of Lefevbre's book (on pages 81, 82), which we also discussed in class.

Problem 5. This problem is a continuation of Problem 4.

(a) **[Food for Thought only!]** Consider the closed and irreducible set C_1 which consists of the recurrent states 1 and 2. Directly from the transition probabilities find the probabilities $\rho_{ij}^{(n)}$ of visiting state j for the first time in exactly n steps starting from state i for all possible i and j in C_1 (draw a simple diagram with these two states and think about the number of ways the first returns/visits can occur).

Solution: All the values of $\rho_{ij}^{(n)}$ for $i, j \in \{1, 2\}$ are computed in on page 82 of Lefevbre's book. You do *not* need to reproduce those computations in your homework, but I expect you to understand completely Lefevbre's arguments.

(b) Use the values you obtained to compute the probabilities f_{ij} of eventually visiting state j starting from state i for all i and j in C_1. Are you surprised by the results for f_{ij}? Explain why (or why not).

(c) Find $\rho_{33}^{(n)}$ and f_{33} for the only state in the set C_2. (Please explain briefly your reasoning.) Answer the same questions as in part (b).

(d) Compute the values of $\rho_{54}^{(n)}$ and $\rho_{55}^{(n)}$. (Please explain briefly your reasoning.)

(e) Compute the values of f_{54} and f_{55}. Discuss your finding in the light of the general theory.

Food for Thought Problem 1. Let Y be a discrete RV taking values y_m, where the index m runs over a discrete set (finite or infinite). For a given m, consider the pre-image of y_m, i.e., the set

$$Y^{-1}(y_m) = \{\omega \in \Omega \,|\, Y(\omega) = y_m\} \ , \quad \text{which is often written as } \{Y = y_m\} \ . \tag{3}$$

(a) Convince yourself that when k runs over all possible values, the sets $\{Y = y_m\}$ form a partition of the sample space Ω, i.e.,

$$\bigcup_m Y^{-1}(y_m) = \Omega \ , \qquad Y^{-1}(y_m) \cap Y^{-1}(y_n) = \varnothing \ \text{ for } m \neq n \ . \tag{4}$$

(b) Let $A \subset \Omega$ be an event. Recall that the *indicator function*, $I_A : \Omega \to \mathbb{R}$, of an event A is defined as

$$I_A(\omega) = \begin{cases} 0 & \text{if } \omega \notin A \ , \\ 1 & \text{if } \omega \in A \ . \end{cases}$$

If $I_A^{-1} : \{0,1\} \to \Omega$ stands for the inverse of I_A (of course, I_A^{-1} can take only the numbers 0 or 1 as an argument), then obviously

$$I_A^{-1}(0) = \{\omega \in \Omega \,:\, I_A(\omega) = 0\} = A^c \ ,$$
$$I_A^{-1}(1) = \{\omega \in \Omega \,:\, I_A(\omega) = 1\} = A \ ,$$

and

$$\begin{aligned} \mathbb{P}(I_A^{-1}(0)) &= \mathbb{P}(\{\omega \in \Omega \,:\, I_A(\omega) = 0\}) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A) \ , \\ \mathbb{P}(I_A^{-1}(1)) &= \mathbb{P}(\{\omega \in \Omega \,:\, I_A(\omega) = 1\}) = \mathbb{P}(A) \ . \end{aligned} \tag{5}$$

For a given m, consider the indicator function $I_{Y^{-1}(y_m)} : \Omega \to \{0,1\}$ of the set $Y^{-1}(y_m)$. This is a discrete RV variable satisfying

$$I_{Y^{-1}(y_m)}(\omega) = \begin{cases} 0 & \text{if } \ \omega \notin Y^{-1}(y_m) \,, \text{ i.e., } \ \omega \notin Y^{-1}(y_m) \ , \\ 1 & \text{if } \ \omega \in Y^{-1}(y_m) \,, \text{ i.e., } \ \omega \in Y^{-1}(y_m) \ . \end{cases}$$

Convince yourself that the RV Y can be written as a linear combination of the indicator functions of the sets $Y^{-1}(y_m)$ as m runs over all allowed values, as follows:

$$Y = \sum_m y_m \, I_{Y^{-1}(y_m)} \tag{6}$$

(simply note that $I_{Y^{-1}(y_m)}(\omega)$ is equal to 1 exactly if $Y(\omega) = y_m$).

(c) Convince yourself that the p.m.f.

$$p_{I_A}(y) = \mathbb{P}(I_A = y) = \mathbb{P}(I_A^{-1}(y)) \qquad \text{for } \ y \in \{0,1\}$$

of the indicator function I_A is equal to (recall (5))

$$p_{I_A}(y) = \mathbb{P}(I_A^{-1}(y)) = \begin{cases} \mathbb{P}(A^c) = 1 - \mathbb{P}(A) & \text{if } \ y = 0 \ , \\ \mathbb{P}(A) & \text{if } \ y = 1 \ . \end{cases} \tag{7}$$

(d) Directly from the definition of expectation and from (7), show that

$$\mathbb{E}[I_A] = \mathbb{P}(A) \ . \tag{8}$$

(e) Take expectation of both sides of (6) and use the linearity property of expectation (i.e., the fact that $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$) and (8) to obtain

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_m y_m \, I_{Y^{-1}(y_m)}\right] = \sum_m y_m \, \mathbb{E}\left[I_{Y^{-1}(y_m)}\right] = \sum_m y_m \, \mathbb{P}(Y^{-1}(y_m)) = \sum_m y_m \, p_Y(y_m) \ ,$$

which coincides with the definition of $\mathbb{E}[Y]$, as it should.