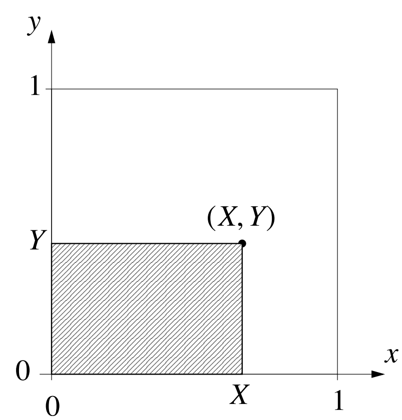
MATH 5763 Homework 3 Due Tue, 9/15/2015, in class!

Problem 1. Let X and Y be independent random variables, each with distribution Uniform$(0,1)$. Then the point with coordinates (X,Y) is a random vector that is uniformly distributed in the unit square, i.e., its probability density function is

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1\ , \quad 0 \le y \le 1\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Let A be the random variable equal to the area of the rectangle with vortices at the points $(0,0)$, $(0,Y)$, $(X,0)$, and (X,Y) – see the figure. Clearly, A is a continuous random variable taking values in the interval $[0,1]$.



(a) In the (x,y) plane, sketch the domain determined by the inequalities $0 \le x \le 1$, $0 \le y \le 1$, $xy \le a$ (where a is a value between 0 and 1).

(b) Show that the cumulative distribution function of A,

$$F_A(a) = \mathbb{P}(A \le a) = \mathbb{P}(XY \le a) = \iint_{xy \le a} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y\ ,$$

is given by

$$F_A(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0]\ , \\ a\,(1 - \ln a) & \text{if } a \in (0, 1]\ , \\ 1 & \text{if } a \in [1, \infty)\ . \end{cases}$$

Hint: What you drew in part (a) will be useful.

(c) Find the probability density function, $f_A(a)$, of A. Be sure to specify $f_A(a)$ for all values of a.

(d) Determine the expected value $\mathbb{E}[A]$ of the area A of the random square with sides of length X and Y.

Problem 2. Let A and B be independent events in the sample space Ω, and let I_A and I_B be the corresponding indicator random variables:

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \ , \\ 0 & \text{if } \omega \notin A \ . \end{cases}$$

Express the following indicator random variables in terms of I_A and I_B:

(a) I_{A^c}; (b) $I_{A \cap B}$; (c) $I_{A \cup B}$.

In each case, explain briefly your reasoning.

Hint: The easiest way to solve this problem is to come up with some guess and then check that the guess was correct.

Problem 3. Let X and Y be independent Poisson random variables with respective parameters λ and μ, i.e.,

$$p_X(k) = \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!} \ , \quad p_Y(k) = \mathrm{e}^{-\mu} \frac{\mu^k}{k!} \ , \qquad k = 0, 1, 2, \dots \ .$$

Show that:

(a) $X + Y$ is Poisson with parameter $\lambda + \mu$;

(b) the conditional distribution of X given that $X + Y = n$ is binomial, and find its parameters.

Hint: (b) Note that $\{X = k\} \cap \{X + Y = n\} = \{X = k\} \cap \{Y = n - k\}$, so that, by the independence of X and Y, $\mathbb{P}(\{X = k\} \cap \{X + Y = n\}) = \mathbb{P}(\{X = k\} \cap \{Y = n - k\}) = \mathbb{P}(X = k)\,\mathbb{P}(Y = n - k)$.

Problem 4. Let X be a stationary discrete-time discrete-state space Markov chain with state space S consisting of two states, 0 and 1, and let the 1-step transition probability matrix of the stochastic process be

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix} , \tag{1}$$

where $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$. Assume that you do not know the exact value of the initial value X_0 of the MC X, but you know that

$$\mathbb{P}(X_0 = 0) = \frac{1}{5} \ . \tag{2}$$

(a) Find the p.m.f. p_{X_0} of the initial state X_0 of the MC.

(b) Find $\mathbb{E}[X_0]$ and $\operatorname{Var} X_0$.

(c) Find the p.m.f. p_{X_1} of the state X_1 of the MC at time 1.

(d) Find $\mathbb{E}[X_1]$.

(e) Find the p.m.f. p_{X_2} of the state X_2 of the MC at time 2.

Problem 5. Consider the MC from Problem 4. Let

$$\rho_{ij}^{(n)} := \mathbb{P}\left(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j | X_0 = i\right) \ , \qquad n \geq 1 \ , \quad i, j \in \{0, 1\}$$

be the probability of moving to state j, from the initial state i, *for the first time* at the nth transition. Recall the relations

$$p_{ij}^{(n)} = \sum_{k=1}^{n} \rho_{ij}^{(k)} \, p_{jj}^{(n-k)} \ , \tag{3}$$

where $p_{ij}^{(n)}$ are the entries of the n-step transition probability matrix $\mathbf{P}^{(n)}$. Assume (quite reasonably) that $\mathbf{P}^{(0)} = \mathbf{I}$ (the identity matrix).

(a) Compute explicitly $\mathbf{P}^{(0)}$, $\mathbf{P}^{(1)}$, $\mathbf{P}^{(2)}$, and $\mathbf{P}^{(3)}$.

(b) Use (3) to compute the value of $\rho_{00}^{(1)}$.

(c) Use (3) and the value of $\rho_{00}^{(1)}$ (found in (b)) to compute the value of $\rho_{00}^{(2)}$.

(d) Use (3) and the values of $\rho_{00}^{(1)}$ and $\rho_{00}^{(2)}$ (found in (b) and (c)) to compute the value of $\rho_{00}^{(3)}$.

Food for Thought Problem 1. Prove the recursive relation (3).

Solution: Let $i \in \{0, 1\}$, $j \in \{0, 1\}$, and $n \geq 1$ be fixed. Notice that the events

$$A_k := \{ \text{ the MC reaches state } j \text{ for the first time in exactly } k \text{ steps } \} \ .$$

Clearly, for a given n, the time k when the MC reaches state j for the first time (starting from state i) can take values 1, 2, …, n. It is obvious that the events A_k form a partition of the sample space Ω (why?):

$$\bigcup_{k=1}^{n} A_k = \Omega \ , \qquad A_k \cap A_{k'} = \varnothing \ \text{ for } k \neq k' \ . \tag{4}$$

Note that

$$\rho_{ij}^{(k)} = \mathbb{P}(A_k \,|\, X_0 = i) \ . \tag{5}$$

Using (4) and (5), we obtain

$$
\begin{aligned}
p_{ij}^{(n)} &= \mathbb{P}\big(X_n = j \,\big|\, X_0 = i\big) \\
&= \mathbb{P}\big(\{X_n = j\} \cap \Omega \,\big|\, X_0 = i\big) \\
&= \mathbb{P}\left(\{X_n = j\} \cap \bigcup_{k=1}^{n} A_k \,\middle|\, X_0 = i\right) \\
&= \mathbb{P}\left(\bigcup_{k=1}^{n} \big(\{X_n = j\} \cap A_k\big) \,\middle|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(\{X_n = j\} \cap A_k \,\middle|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \frac{\mathbb{P}\left(\{X_n = j\} \cap A_k \cap \{X_0 = i\}\right)}{\mathbb{P}(X_0 = i)} \\
&= \sum_{k=1}^{n} \frac{\mathbb{P}\left(\{X_n = j\} \cap A_k \cap \{X_0 = i\}\right)}{\mathbb{P}\left(A_k \cap \{X_0 = i\}\right)} \; \frac{\mathbb{P}\left(A_k \cap \{X_0 = i\}\right)}{\mathbb{P}(X_0 = i)} \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_n = j \,|\, A_k \cap \{X_0 = i\}\right) \; \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_n = j \,|\, A_k\right) \; \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_n = j \,|\, X_k = j\right) \; \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_{n-k} = j \,|\, X_0 = j\right) \; \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} p_{jj}^{(n-k)} \, \rho_{ij}^{(k)} \; .
\end{aligned}
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