MATH 5483 Homework 3 Due Thu, Feb 7

Problem 1.

(a) Prove that any increasing sequence of real numbers that is bounded above will converge.

(b) Prove that any convergent sequence of real numbers is bounded.

Problem 2. As you know, one way to approximate a (differentiable) function $f(x)$ is to replace it by its tangent line at some point of interest. This type of approximation, however, works well only near that point, and can be very inaccurate over an entire *interval.*

One way to approximate a real-valued function $f(x)$ by a (real-valued) linear function, $g(x) = ax + b$, over an interval $[x_0, x_1]$ is to choose values of the real constants a and b that minimize the value of the L^2-error $\|f - g\|_{L^2([x_0,x_1])}$.

Determine the real constants a and b that minimize the $L^2([0, 1])$ error for the function $f(x) = x^2$ over the interval $[0, 1]$. You may find the following formula useful:

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma \ .$$

Problem 3. Recall that, given $a > 0$ and $b \in \mathbb{R}$, the *dilation operator* D_a and the *translation operator* T_b are defined on functions in $L^2(\mathbb{R})$ by

$$(D_a f)(x) := a^{1/2} f(ax) \ , \qquad (T_b f)(x) := f(x - b) \ .$$

(a) If $f(x) = \mathrm{e}^{-x^2}$, find $D_a T_b f(x)$ and $T_b D_a f(x)$.

(b) Show that $\langle f, D_a g\rangle = \langle D_{1/a} f, g\rangle \ \forall \ f, g \in L^2(\mathbb{R})$ and $a > 0$.

(c) Show that $\langle f, T_b g\rangle = \langle T_{-b} f, g\rangle \ \forall \ f, g \in L^2(\mathbb{R})$ and $b \in \mathbb{R}$.

(d) Show that $\langle D_a f, D_a g\rangle = \langle f, g\rangle$ and $\langle T_b f, T_b g\rangle = \langle f, g\rangle \ \forall \ f, g \in L^2(\mathbb{R})$, $a > 0$, $b \in \mathbb{R}$. (Operators like D_a and T_b that preserve the inner product are called *isometries.*)

Problem 4. Let the function $f \in L^2(\mathbb{R})$ be defined by $f(x) = x^2 \chi_{[0,1]}(x)$, and let P_j and Q_j be the approximation operator and the detail operator at scale j as defined in Sec. 5.5 of the book.

(a) Compute and draw $P_0 f(x)$.

(b) Compute and draw $P_1 f(x)$ and $Q_0 f(x)$.

(c) Compute and draw $P_2 f(x)$ and $Q_1 f(x)$.

Problem 5. Prove that if $f \in C_c^0(\mathbb{R})$ and P_j is the scale j approximation operator, then

$$\lim_{j\to\infty} \|P_{-j} f\|_{L^2(\mathbb{R})} = 0 \ .$$

Hint: Use Minkowski's inequality and the fact that f is compactly supported.