MATH 5463 Homework 3 Due Thu, Feb 19

Problems 34, 35, 42(modified–see below) from Section 3.5 of the book.

Hint to Problem 34: Let $\Omega = \{(x, y) : a \leq x \leq y \leq b\}$, and compute $(\mu_F \times \mu_G)(\Omega)$ in two ways, as in the proof of Theorem 3.36, to show that

$$\int_{[a,b]} F(x)\,\mathrm{d}G(x) + \int_{[a,b]} G(x-)\,\mathrm{d}F(x) = F(b)G(b) - F(a-)G(a-)\ .$$

Then, swap the places of F and G in this equality to obtain

$$\int_{[a,b]} G(x)\,\mathrm{d}F(x) + \int_{[a,b]} F(x-)\,\mathrm{d}G(x) = F(b)G(b) - F(a-)G(a-)\ ,$$

and derive the desired result from these two equalities.

Hint to Problem 35: You first have to prove that if F and G are in then $FG \in \mathrm{AC}([a, b])$. This is easy, recalling that $F \in \mathrm{AC}([a, b])$ implies that $F \in \mathrm{BV}([a, b])$, which, in turn, implies that F is bounded (why?). Having established that $FG \in \mathrm{AC}([a, b])$, you can use the Fundamental Theorem of Calculus for Lebesgue Integrals to obtain the desired formula.

Additions, remarks and hints to Problem 42:

(a) Think geometrically. To prove that the desired inequality follows from the convexity of F, you can prove the inequalities

$$\frac{F(t) - F(s)}{t - s} \leq \frac{F(t') - F(s)}{t' - s} \leq \frac{F(t') - F(s')}{t' - s'}$$

(for $s \leq s' < t'$ and $s < t \leq t'$). If you prove the left inequality, the right one is completely analogous (draw a picture representing these inequalities). To prove the left inequality, you can take the defining property of the convex functions in the form $F(\lambda s + (1 - \lambda)t') \leq \lambda F(s) + (1 - \lambda)F(t')$, with $\lambda = \frac{t'-s'}{t'-s} \in (0, 1)$ (why did I choose this value of λ?). This same idea will help you prove the converse (i.e., that these inequalities imply the convexity of the function).

(b) *In this part prove only that the convexity of F implies the absolute continuity of F on compact subintervals, as well as that F' is non-decreasing (wherever it is defined).*

To show that convexity implies that F' is non-decreasing (wherever F' is defined), think of s and t' in part (a) as fixed values ($s < t'$), and take the limits $t \downarrow s$ and $s' \uparrow t'$ in the inequality from part (a) (if these limits exist).

(c) Again, think geometrically about the meaning of $\frac{F(t)-F(t_0)}{t-t_0}$. What would β be if F was differentiable at t_0? (Keep in mind that the absolute continuity on compact subintervals proved in part (a) implies that F is differentiable almost everywhere, so that thinking about derivatives is not too unrealistic here.)

(e) Let $X = \{a_1, \ldots, a_n\}$ be a finite set, μ be a measure on X defined by $\mu(\{a_j\}) = \frac{1}{n}$ for each $j \in \{1, \ldots, n\}$ (so that $\mu(X) = 1$, as it should), $g(a_j) = \ln x_j$, where x_j ($j \in \{1, \ldots, n\}$) are some positive numbers, and $F : \mathbb{R} \to \mathbb{R}$ be the (obviously convex) exponential function: $F(x) = \mathrm{e}^x$. Apply the inequality from part (d) in this setting to obtain the well-known ineqality between the arithmetic mean and the geometric mean of n positive numbers,

$$(x_1 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n} \; .$$

Additional problem 1. Show by example that, if r and s are two distinct numbers in $[1, \infty)$, then $L^r(\mathbb{R}, \mathcal{B}, m) \not\subseteq L^s(\mathbb{R}, \mathcal{B}, m)$ and $L^s(\mathbb{R}, \mathcal{B}, m) \not\subseteq L^r(\mathbb{R}, \mathcal{B}, m)$.

Hint: Think of a simple example.

Additional problem 2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space (i.e., $\mu(X) < \infty$). Assume that $\mu(X) = 1$, i.e., μ is a probability measure (this is always possible to achieve for finite measures). In this problem you will show that if $0 < r < s$, then $L^s(X, \mathcal{M}, \mu) \subseteq L^r(X, \mathcal{M}, \mu)$, and this inclusion is strict (i.e, $L^s(X, \mathcal{M}, \mu) \neq L^r(X, \mathcal{M}, \mu)$).

(a) Recall the Jensen's inequality from Problem 3.5/42(d):

$$F\left(\int g \, \mathrm{d}\mu\right) \leq \int F \circ g \, \mathrm{d}\mu \; ,$$

where μ is a probability measure, $g : X \to (a, b)$ is in $L^1(X, \mathcal{M}, \mu)$, and $F : (a, b) \to \mathbb{R}$ is a convex function. Let f be an arbitrary function in $L^s(X, \mathcal{M}, \mu)$. Set $g = |f|^r$ and $F : \mathbb{R} \to \mathbb{R} : x \to |x|^{s/r}$; note that, since $\frac{s}{r} > 1$, F is indeed a convex function. Apply Jensen's inequality to obtain that $\|f\|_r \leq \|f\|_s$. What does this imply about the spaces $L^s(X, \mathcal{M}, \mu)$ and $L^r(X, \mathcal{M}, \mu)$?

(b) Let $(X, \mathcal{M}, \mu) = ([0, 1], \mathcal{B}_{[0,1]}, m_{[0,1]})$, where $m_{[0,1]}$ be the restriction of the Lebesgue measure to $\mathcal{B}_{[0,1]}$. If $0 < r < s$, find an example of a function $f : [0, 1] \to \mathbb{R}$ that is in $L^r([0, 1], \mathcal{B}_{[0,1]}, m_{[0,1]})$, but not in $L^s([0, 1], \mathcal{B}_{[0,1]}, m_{[0,1]})$.

(c) Finally, give an alternative derivation of the result that you already obtained in (a) by using Hölder's inequality, $\|\phi\psi\|_1 \leq \|\phi\|_p \|\psi\|_q$, where p and q are conjugate exponents: $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Hint: Try setting $\phi = |f|^r$, $p = \frac{s}{r} > 1$. What should you take for ψ? (In the choice of ψ you will see that the requirement that μ be finite is crucial.)

Remark: It is not so surprising that we could use Hölder's inequality to derive a result that we previously derived by using Jensen's inequality. In fact, Lemma 6.1, which was the crucial ingredient in the proof of Hölder's inequality follows directly from the fact that the function $x \mapsto \mathrm{e}^x$ is convex (how?).