

Problem 18 from Section 1.3 of the book.

Hint to part (b): In the proof of the “if” part, use what you proved in (a) for a sequence of sets $\{A_j\}_{j=1}^\infty \subset \mathcal{A}_\sigma$ satisfying $\mu^*(A_j) \leq \mu^*(E) + \epsilon_j$, where $\{\epsilon_j\}_{j=1}^\infty$ is a decreasing sequence of positive numbers. The easiest way to prove the “only if” part is to use the Carathéodory’s Theorem.

Additional problem 1. Let $\mu_0 : \mathcal{A} \rightarrow [0, \infty]$ be a finite premeasure defined on the algebra $\mathcal{A} \subset \mathcal{P}(X)$, and let $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ be the outer measure induced by μ_0 ,

$$\mu^*(E) := \inf \left\{ \sum_{j=1}^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \bigcup_{j=1}^\infty A_j \right\} .$$

Define the *inner measure* $\mu_* : \mathcal{P}(X) \rightarrow [0, \infty]$ by $\mu_*(E) := \mu_0(X) - \mu^*(E^c)$.

- (a) Prove that $\mu_*(E) \leq \mu^*(E)$ for any $E \subset X$.
- (b) Prove that if $E \in \mathcal{A}$, then $\mu_*(E) = \mu^*(E)$.
- (c) Demonstrate that μ_* is monotone, i.e., if $E \subset F$, then $\mu_*(E) \leq \mu_*(F)$.
- (d) In the notations of Problem 1.3/18, show that E is μ^* -measurable if and only if there exists a set $C \in \mathcal{A}_{\sigma\delta}$ such that $E^c \subset C$ and $\mu^*(C \setminus E^c) = 0$.
- (e) Use the fact proved part (d) to show that if $\mu_*(E) = \mu^*(E)$, then E is μ^* -measurable.
- (f) Finally, show that if E is μ^* -measurable, then $\mu_*(E) = \mu^*(E)$, which together with the result of part (e) implies that E is μ^* -measurable iff $\mu_*(E) = \mu^*(E)$.

Additional problem 2. Let (X, \mathcal{M}, μ) be a measure space. Show that if $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$ and $\sum_{j=1}^\infty \mu(E_j) < \infty$, then $\mu(\limsup E_j) = 0$ (this is the so-called *(first) Borel-Cantelli lemma*, very important in Probability Theory).

Additional problem 3. The *Cantor set* is a subset of $[0, 1]$ defined as follows. First remove the open middle third, $(\frac{1}{3}, \frac{2}{3})$, from $[0, 1]$. Then remove the open middle thirds, $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, from the remaining two intervals, $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. After that remove the middle thirds, $(\frac{1}{27}, \frac{2}{27})$, $(\frac{7}{27}, \frac{8}{27})$, $(\frac{19}{27}, \frac{20}{27})$ and $(\frac{25}{27}, \frac{26}{27})$, from the remaining four intervals, $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$. Keep removing the middle third of the remaining intervals infinitely many times. Call the remaining set the *(middle-third) Cantor set*, and denote it by C .

- (a) One can easily see that each number in C has base-3 decimal expansion of the form

$$x = (0.a_1a_2a_3 \dots)_3 := \sum_{j=1}^{\infty} \frac{a_j}{3^j},$$

with a_j equal to 0 or 2. Such expansions are unique except for numbers of the form $(0.a_1a_2 \dots a_k00000 \dots)_3$ which can also be written as $(0.\tilde{a}_1\tilde{a}_2 \dots \tilde{a}_k2222 \dots)_3$; if this is the case, we use the latter expansion. Construct a surjective function $f : C \rightarrow [0, 1]$; the existence of such function implies that C has a cardinality of the continuum.

Hint: For $x = (0.a_1a_2a_3 \dots)_3$, write $f(x)$ in base-2 decimal expansion,

$$f(x) = (0.b_1b_2b_3 \dots)_2 := \sum_{j=1}^{\infty} \frac{b_j}{2^j},$$

with b_j equal to 0 or 1.

- (b) Show that C is compact.
- (c) Prove that C is nowhere dense, i.e., the closure of C has an empty interior (see pp. 13–14 of the book).
- (d) Prove that C is totally disconnected, i.e., the only connected subsets of C are single points. (Show that if $x, y \in C$ with $x < y$, there exists $z \notin C$ such that $x < z < y$.)
- (e) Show that C has no isolated points.
- (f) Is C a Borel set? Justify your answer.