MATH 4513 Homework 3 Due Fri, Sep 18, 2015

Problem 1. [Thinking simply, yet again]
The total resistance of two resistors, R_1 and R_2, in series is

$$R_{\text{in series}} = R_1 + R_2 \ ,$$

while their total resistance in parallel is given by

$$\frac{1}{R_{\text{in parallel}}} = \frac{1}{R_1} + \frac{1}{R_2} \ .$$

Use these facts to find the total resistance between the points A and D (the leftmost points) of the infinite chain of resistors drawn in Figure 1. The resistance of each resistor is $R = 1\,\text{Ohm}$. You can find the total resistance very simply, if you think similarly to Problem 1 from Homework 2.

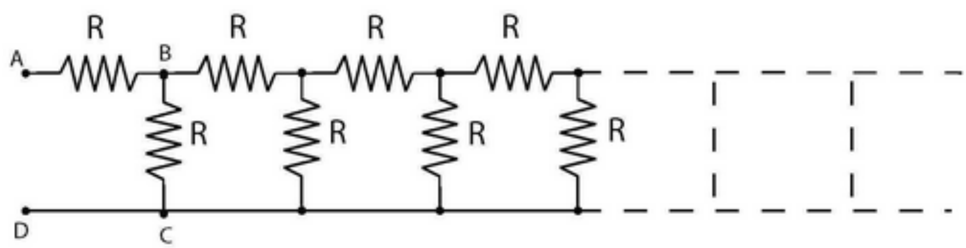

Figure 1: Infinite chain of resistors.

Problem 2. [Multiplicity of a zero]
Recall that we say that x_0 is a *zero of multiplicity* m (for some $m \in \mathbb{N}$) of a real-valued function f if f can be written as

$$f(x) = (x - x_0)^m \, q(x) \ ,$$

where $q(x)$ is a function such that $g(x_0) \neq 0$.

(a) The Taylor expansion of a function $\phi(x)$ around $x_0 = 3$ be

$$\phi(x) = \frac{1}{2}(x-3)^2 - \frac{1}{3}(x-3)^4 + \frac{1}{3}(x-3)^6 - \frac{23}{90}(x-3)^8 + \frac{181}{720}(x-3)^{10} \cdots \ .$$

What is the multiplicity m of 3 as a zero of $\phi(x)$? Explain why.
Hint: The easiest way to do this is to use the very definition of multiplicity.

(b) Directly from the definition of multiplicity of a zero, show that if x_0 is a zero of multiplicity m_1 of the function $f(x)$, and a zero of multiplicity m_2 of the function $g(x)$, then it is zero of multiplicity $m_1 + m_2$ of the product $f(x)\, g(x)$.

(c) Show that $x_0 = 0$ is a zero of multiplicity 3 of the function $f(x) = x - \sin x$.

(d) Use your results from parts (b) and (c) to find the multiplicity of the zero $x_0 = 0$ of the function

$$h(x) = x^4 \left(x - \sin x\right)^2 \ .$$

(e) If you know that a function $f(x)$ has a zero of multiplicity $m \geq 2$ at the point x_0, what can you say about the multiplicity of x_0 as zero of the derivative $f'(x)$? Justify your answer briefly.

Problem 3. [Subcritical pitchfork bifurcation]

(a) Consider the first-order ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + bx^3 - cx^5 \ , \tag{1}$$

where a, b, and c are parameters, satisfying $b > 0$, $c > 0$. Show that two of these parameters can be eliminated by an appropriate change of variables, so that the ODE can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \alpha y + y^3 - y^5 =: f(y) \ . \tag{2}$$

Write down explicitly the change of variables, and show that it brings equation (1) to the form (2).

(b) In the rest of the problem you will study the bifurcation occurring at $\alpha = 0$, and will show that it is a subcritical pitchfork bifurcation.

Show that for $\boxed{\alpha > 0}$, the ODE (2) has three FPs: one FP equal to 0, and the other two are "far from 0" (namely, near -1 and 1 for small positive values of α). Please write the exact expressions for all the FPs.

(c) For $\boxed{\alpha > 0}$, sketch roughly the graph of $f(y) = \alpha y + y^3 - y^5$ (explain why it looks like this), find $f'(0)$, and indicate the stability of all three fixed points as usual (full circles for the stable FPs and empty circles for the unstable FPs); there is no need to do long computations, just look at the graph.

(d) Show that for $\boxed{\alpha < 0}$, the ODE (2) has five FPs: one FP equal to 0, two "far from 0" FPs (near -1 and 1 for small values of α), and two "near 0" FPs (approaching to 0 when $\alpha \to 0^-$). Please write the exact expressions for all the FPs in this case.

(e) For $\boxed{\alpha < 0}$, find $f'(0)$ and use this information (and the general form of $f(y)$) to draw a rough sketch of $f(y)$. Determine the stability of each of the FPs, draw the "direction of motion" denote the stable and the unstable FPs in the graph as usual.

(f) For $\boxed{\alpha < 0}$, use that the Taylor expansion of the function $g(\xi) = (1+\xi)^\beta$ for β not equal to a non-negative integer is

$$(1+\xi)^\beta = 1 + \beta\xi + \frac{\beta(\beta-1)}{2}\xi^2 + \frac{\beta(\beta-1)(\beta-2)}{6}\xi^3 + \cdots \approx 1 + \beta\xi$$

to show that the FPs near 0 are approximately $\pm\sqrt{-\alpha}$.

(g) In the (α, x^*)-plane, sketch the bifurcation diagram of the ODE (2). Pay special attention to the FPs near 0.

Problem 4. [... and a little bit of hard work]

This problem does <u>not</u> need to be turned in, but you are expected to understand it completely!

In Problem 4 of Homework 2 you studied the bifurcation in a logistic equation with linear harvesting, described by the 2-parameter family of ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = y(1-y) - (a+by) =: f(y) \ , \tag{3}$$

where a and b are positive parameters.

In that problem you rewrote the condition $f(y^*) = y^*(1-y^*) - (a+by^*) = 0$ for a FP of (3) in the form $g(y^*) = h(y^*)$ with $g(y) = y(1-y)$ and $h(y) = a + by$. Then you plotted the graphs of $g(y)$ and $h(y)$ together, and used that for a bifurcation to occur, they should be tangent at the point y^*, which can be written as

$$\begin{aligned} g(y^*) &= h(y^*) \qquad &&\text{(the graphs have a common point)} \ , \\ g'(y^*) &= h'(y^*) \qquad &&\text{(the graphs have the same tangent line for } y^*) \ . \end{aligned}$$

This can be rewritten as

$$\begin{aligned} y^*(1-y^*) &= a + by^* \ , \\ 1 - 2y^* &= b \ . \end{aligned} \tag{4}$$

After excluding y^* from (4), we obtain the relation between a and b for a saddle-node bifurcation to occur:

$$a = \frac{(1-b)^2}{4} \ . \tag{5}$$

In Figure 2, you see the graph of $g(y)$ and several graphs of $h(y)$ for several pairs (a, b) satisfying (5) (namely, for $(a,b) = (\frac{1}{25}, \frac{3}{5})$, $(\frac{1}{9}, \frac{1}{3})$, $(\frac{1}{5}, 1 - \frac{2}{\sqrt{5}})$, and $(\frac{1}{4}, 0)$).

In Figure 3, we plotted the domains in the (a, b)-plane where the 2-parameter family (3) has different number of FPs (one FP on the parabola (5), two FPs to the left of the parabola, and no FPs to the right of the parabola).

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In[233]:= Plot[{y (1 - y), 1 / 25 + 3 / 5 * y, 1 / 9 + 1 / 3 * y, 1 / 4 + 0 * y, 1 / 5 + (1 - 2 / Sqrt[5]) * y},
  {y, 0, 1}, PlotRange → {{0, 1}, {0, 0.35}}]
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Out[233]=

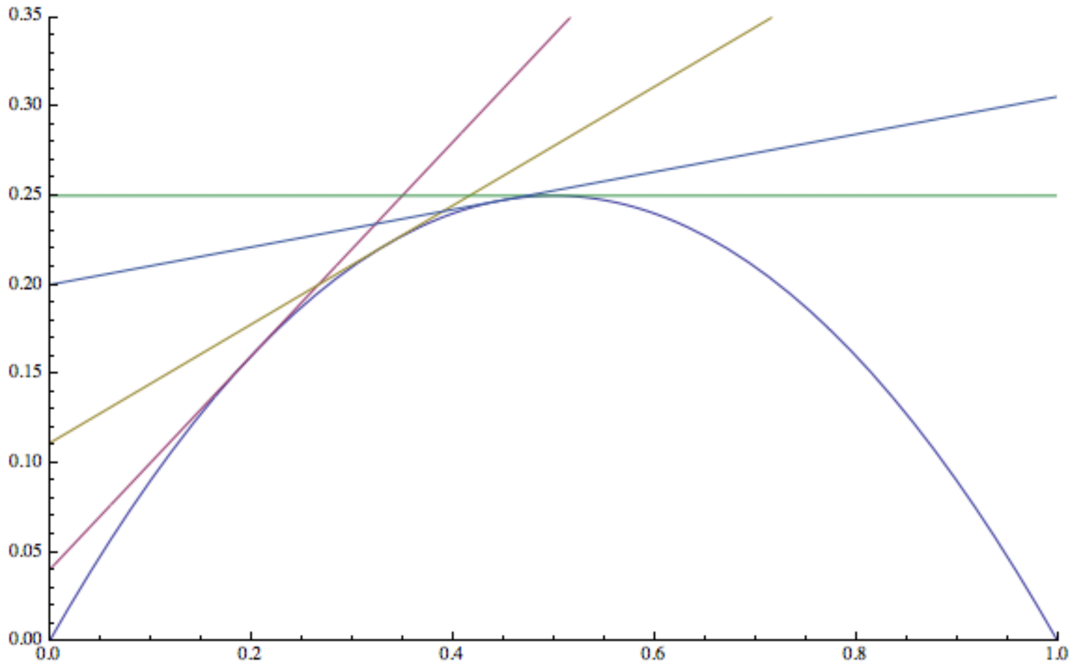

Figure 2: Graphs of $g(y) = y(1-y)$ and $h(y) = a + by$, for pairs (a, b) satisfying (5).

(a) Take $a = 2$, $b = 3$. What does the general theory (i.e., the considerations above) predict about the number of FPs of the family (3) in this case?

(b) For $a = 2$, $b = 3$, find the general solution of (3), and the particular solution with initial condition $y(0) = y_0$, where y_0 is some positive constant.

Hint: The result is $y(\tau) = \tan\bigl(\arctan(y_0+1)-\tau\bigr)-1$, but I want to see your calculations.

(c) Use the concrete expression for the solution of the initial value problem that you obtained in part (b), to show that $y(\tau)$ behaves as predicted by the general theory.

(d) What does the general theory predict for $a = b = \frac{1}{6}$? What are the position(s) and the stability of the FP(FPs)?

(e) Find the solution of (3) with $a = b = \frac{1}{6}$ and initial condition $y(0) = y_0 > 0$. You may use (without deriving it) that

$$\int \frac{\mathrm{d}y}{\left(y-\frac{1}{3}\right)\,\left(y-\frac{1}{2}\right)} = 6\ln\left|\frac{6y-3}{6y-2}\right| + C$$

(which I obtained by using the method of partial fractions).

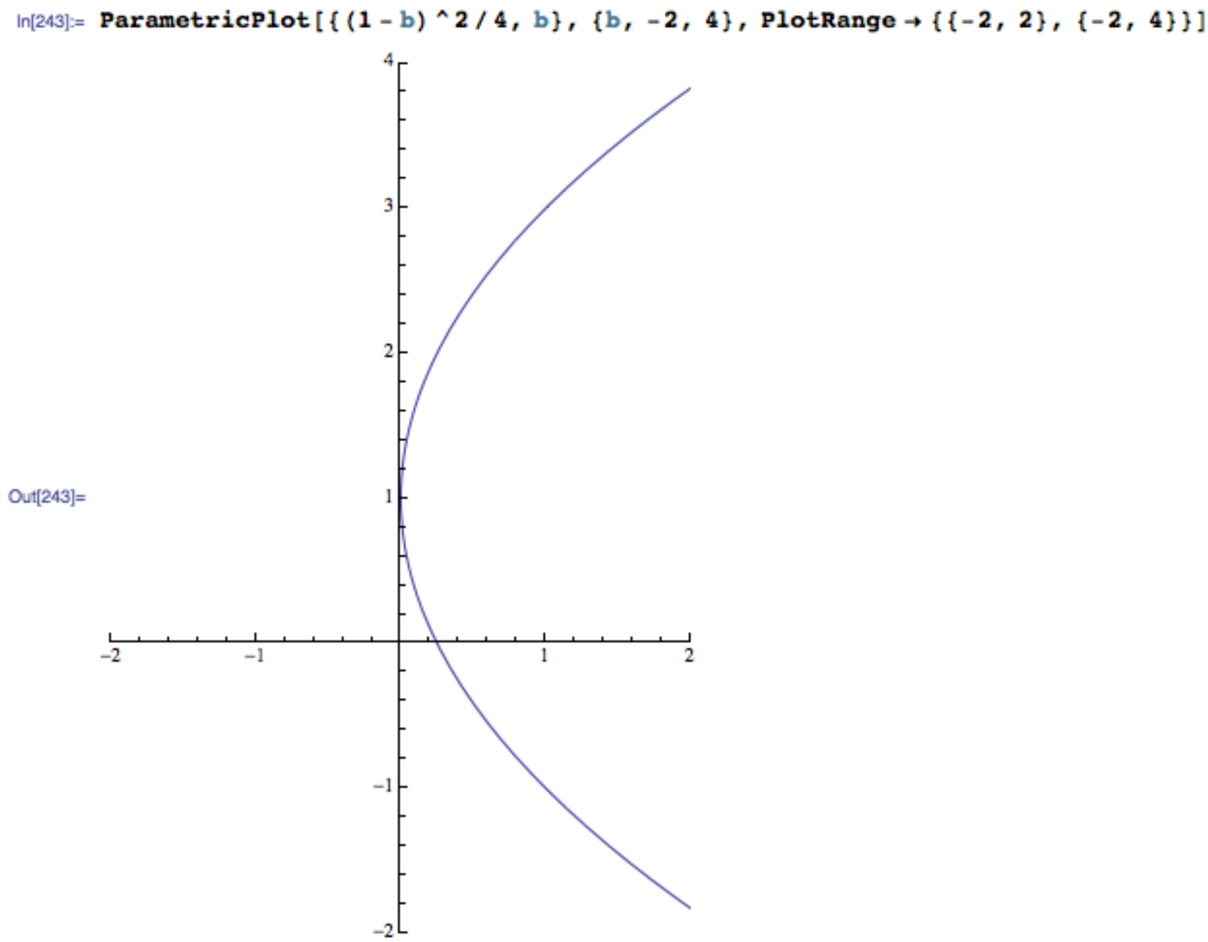

Figure 3: Domains in the (a, b)-plane corresponding to different numbers of FPs of the 2-parameter family (3).