MATH 4193/5103 Homework 3 Due 3/2/17 (Thursday)

Problem 1. [On the importance of thinking simply]

The distance between Norman, OK, and Waco, TX, is 270 miles, measured along I-35, which we assume to be a straight line (quite a realistic assumption). Two big trucks start driving on I-35 towards each other – one from Norman, the the other one from Waco. The one leaving from Norman is moving at 50 mph, the one leaving from Waco is moving at 40 mph.

At the same moment when the trucks start moving, an eagle starts flying from Norman towards Waco, at 70 mph. As soon as the eagle reaches the truck driving to the North, it turns around and starts flying to the North; as soon at it reaches the truck driving to the South, it turns around and starts flying to the South; the eagle keeps doing this until the two trucks meet, flying all the time at a speed of 70 mph.

Find the total distance traveled by the eagle from the beginning of its flight to the moment when the two trucks meet on the highway. Please explain your reasoning in detail. The solution can be very simple or quite complicated, depending on how you approach the problem.

Problem 2. [Saddle-node and subcritical pitchfork in a 1-parameter family]

(a) Consider the 3-parameter family of first-order ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay + by^3 - cy^5 \ , \tag{1}$$

where a, b, and c are parameters, satisfying $b > 0$, $c > 0$. Two of the three parameters can be eliminated by the change of variables

$$y = \sqrt{\frac{b}{c}}\, x \ , \qquad t = \frac{c}{b^2}\, \tau \ .$$

Show that after this change of variables, the 3-parameter family (1) is transformed into the 1-parameter family

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_\mu(x) := \mu x + x^3 - x^5 \ . \tag{2}$$

Express the parameter μ in terms of the parameters a, b, and c in the original equation (1).

(b) The fixed points of the 1-parameter family (2) are the roots of the algebraic equation $f_\mu(x) = 0$, which can be written as

$$f_\mu(x) = -x(x^4 - x^2 - \mu) = 0 \ .$$

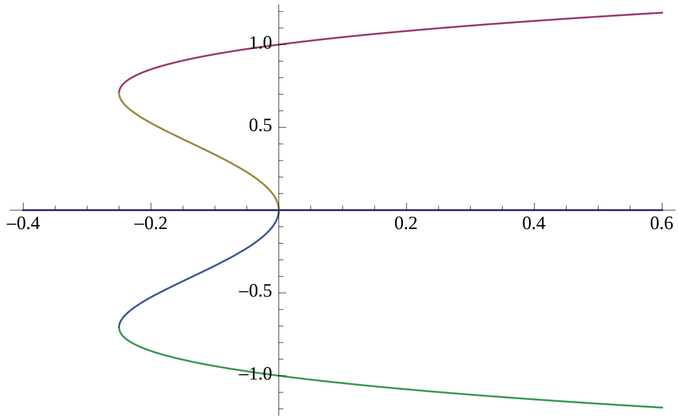
It is clear that $x_0^* = 0$ is always a root of this equation, i.e., a fixed point of (2). The other fixed points of (2) are the roots of the biquadratic equation

$$x^4 - x^2 - \mu = 0 \ ,$$

and are given by

$$x^*_{1,2,3,4} = \pm\sqrt{\frac{1 \pm \sqrt{1+4\mu}}{2}} \ . \tag{3}$$

The graph of the fixed points x^* of (2) as functions of the parameter μ is shown in the figure below (the stability of the fixed points is not indicated). In the rest of this



problem you will identify all the bifurcations that occur in (2).

Using (3), show that for $\boxed{\mu > 0}$, the ODE (2) has exactly three FPs: one FP equal to 0, and the other two are "far from 0" (they are close to -1 and 1 for small positive values of μ). Write the exact expressions for two nonzero FPs.

(c) For $\boxed{\mu > 0}$, find $f'_\mu(0)$, sketch roughly the graph of $f_\mu(x) = \mu x + x^3 - x^5$ (explain why it looks like this), and indicate the stability of all three fixed points as usual (full circles for the stable FPs and empty circles for the unstable FPs); there is no need to do long computations, just look at the graph. What kind of bifurcation does the system undergo at $\mu = 0$?

(d) Using (3), show that for $\boxed{-\frac{1}{4} < \mu < 0}$, the ODE (2) has five FPs: one FP equal to 0, two "far from 0" FPs (near -1 and 1 for small values of μ), and two "near 0" FPs (approaching to 0 when $\mu \to 0^-$). Please write the exact expressions for all the FPs in this case.

(e) For $\boxed{-\frac{1}{4} < \mu < 0}$, find $f'_\mu(0)$ and use this information (and the general form of $f_\mu(x)$) to draw a rough sketch of $f_\mu(x)$. Determine the stability of each of the FPs, draw the "direction of motion" denote the stable and the unstable FPs in the graph as usual.

(f) For $\boxed{\mu < -\frac{1}{4}}$, explain why there is only one fixed point of (2); draw a rough sketch of $f_\mu(x)$, and indicate the stability of this fixed point.

(g) What kind of bifurcation does the system exhibit at $\mu = -\frac{1}{4}$? You can answer this question by just looking at the graphs of $f_\mu(x)$ drawn in the previous part of the problem.

(h) The Taylor expansion of the function $h(z) = (1+z)^\beta$ for β not equal to a non-negative integer is

$$(1+z)^\beta = 1 + \beta z + \frac{\beta(\beta-1)}{2!} z^2 + \frac{\beta(\beta-1)(\beta-2)}{3!} z^3 + \cdots \approx 1 + \beta z \ ;$$

in particular,

$$\sqrt{1+z} = 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 - \frac{5}{128} z^4 + \cdots \ .$$

Use this for $\mu = -\varepsilon$, where ϵ is a very small positive number, to show that the FPs near 0 are approximately $\pm\sqrt{-\mu}$.

(i) In the (μ, x^*)-plane, sketch the bifurcation diagram of the ODE (2), and indicate the stabilities of all the fixed points and all the important values on the axes.

(j) $\boxed{\textbf{[Only if you are taking the class as 5103!]}}$

Write down the Taylor expansion of x^* near the point $(\mu, x^*) = (-\frac{1}{4}, \frac{1}{\sqrt{2}})$ in the bifurcation diagram, as a function of the parameter μ for $\mu = -\frac{1}{4} + \varepsilon$, where ε is a very small positive number.

Problem 3. [Bifurcation in a logistic equation with linear harvesting]

Consider a population $N(t)$ that changes according to the logistic equation and in addition is subjected to a linear harvesting, i.e., in each time interval, a part of the population is removed by harvesting, and the harvesting is assumed to be a linear function of the population at that moment. If α is the reproduction rate of the population, K is the carrying capacity, and A and B are parameters that define the harvesting, the ordinary differential equation governing the evolution of the population is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha N \left(1 - \frac{N}{K}\right) - (A + BN) \ . \tag{4}$$

(a) It looks like the equation (4) has four parameters, but in fact two of them can be eliminated by a change of variables. Change the independent variable t and the dependent variable N to the new "time" τ and "population" x by

$$t = \frac{\tau}{\alpha} \ , \qquad N = Kx \ ,$$

and show that the four-parameter family (4) becomes the two-parameter family

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-x) - (a + bx) \ . \tag{5}$$

Express the new parameters, a and b, in therms of the old parameters α, K, A, and B.

(b) Rewrite the condition $x(1-x)-(a+bx)=0$ for a FP of (5) in the form $f(x)=g_{a,b}(x)$ with $f(x)=x(1-x)$ and $g_{a,b}(x)=a+bx$. Plot the graphs of $f(x)$ and $g_{a,b}(x)$ together, for three cases: when (5) has no FP, when (5) has exactly one FP, and when (5) has two FPs.

(c) Write down the conditions for the equation (5) to have exactly one FP. Solve them to obtain a relation between the parameters a and b.

Hint: Recall that the graphs of $f(x)$ and $g_{a,b}(x)$ must "touch" at a point, which gives you two conditions.

(d) Plot the relation obtained in part (c) in the (a,b) plane, and indicate how many FPs of (5) are there in each region in your plot.