MATH 4163 **Homework 3** **Due Thu, 02/08/2018**

Problem 1. [Diffusion of uniformly moving substance that decays with time]
Let $u(x,t)$ be the concentration of a substance in an infinitely long narrow channel through which water flows in positive x-direction with constant speed c (called the *advection speed*). Since the channel is narrow, we can neglect the variation of the concentration within the cross-section of the channel, and think of u as a function only of the coordinate x along the channel and the time t. Assume also that the substance is decaying exponentially with time (which is exactly the case if it is radioactive or because of chemical reactions). This can be used as a rough model for the propagation of 4-methylcyclohexanemethanol that leaked into the Elk River on January 9, 2014 and contaminated the water in Charleston, West Virginia. The concentration u of the substance is described by the partial differential equation

$$u_t = \alpha^2 u_{xx} - c\, u_x - \gamma\, u \ , \qquad u = u(x,t) \ , \quad x \in \mathbb{R} \ , \quad t > 0 \ , \tag{1}$$

where α, c, and γ are positive constants. The term $\alpha^2 u_{xx}$ in the right-hand side of the PDE corresponds to the diffusion of the substance in the water (which occurs whether or not the water is moving), while γ gives the decay rate (if $T_{1/2}$ is the half-life of a radioactive substance, then you can easily show that $\gamma = \frac{\ln 2}{T_{1/2}}$).

(a) Change the unknown function $u(x,t)$ to the function $v(x,t)$ related to $u(x,t)$ by

$$u(x,t) = \mathrm{e}^{-\gamma t}\, v(x,t) \ .$$

Derive the PDE satisfied by the new function $v(x,t)$.

(b) Change the variables (x,t) in $v(x,t)$ to the new variables $(\widetilde{x},\widetilde{t})$ related to the old ones by

$$\widetilde{x} = x - ct \ , \qquad \widetilde{t} = t \ ,$$

and, correspondingly, change the function $v(x,t)$ to a new function $\widetilde{v}(\widetilde{x},\widetilde{t})$ as usual. Derive the PDE satisfied by $\widetilde{v}(\widetilde{x},\widetilde{t})$.

(c) If the function $\Phi(\widetilde{x},\widetilde{t})$ satisfies the PDE derived in part (b), write down the corresponding solution $u(x,t)$ of (1).

Problem 2. [A boundary value problem for ODEs]
Consider the boundary value problem (BVP)

$$\begin{aligned} &X''(x) - \mu X(x) = 0 \ , \quad x \in [0,L] \ , \\ &X(0) = 0 \ , \quad X'(L) = 0 \ , \end{aligned} \tag{2}$$

where μ is a constant and $L = \text{const} > 0$.

(a) Assume that $\mu > 0$ and set $\mu = \lambda^2$ for some $\lambda > 0$. Find the general solution of (2).

(b) Impose the boundary conditions $X(0) = 0$ and $X'(L) = 0$ on the general solution of (2) found in part (a) and show that the BVP (2) does not have a non-trivial solution (i.e., a solution that is no identically zero on $[0, L]$).

(c) Solve the BVP in the case $\mu = 0$ and show that it has no no non-trivial solution.

(d) Assume that $\mu < 0$ and set $\mu = -\lambda^2$ for some $\lambda > 0$. Find the general solution of (2).

(e) Impose the boundary conditions $X(0) = 0$ and $X'(L) = 0$ on the general solution of (2) found in part (d). Find the condition on the constant λ (and, hence, on the constant μ) under which the BVP (2) has a non-trivial solution on $[0, L]$. Write down the corresponding solutions.

Problem 3. [Differentiation of an integral depending on a parameter]

The solution of the initial value problem for the wave equation

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} , \qquad x \in \mathbb{R} , \quad t \geq 0 , \\ &u(x,0) = 0 , \\ &u_t(x,0) = h(x) \end{aligned}$$

(where c is a positive constant) is given by a particular case of the so-called *D'Alembert's formula*,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(z)\,\mathrm{d}z \tag{3}$$

(you do not need to prove this).

The following formula holds for differentiating an integral whose limits and integrand depend on some parameter α:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\varphi(\alpha)}^{\psi(\alpha)} F(y,\alpha)\,\mathrm{d}y = F(\psi(\alpha),\alpha)\,\psi'(\alpha) - F(\varphi(\alpha),\alpha)\,\varphi'(\alpha) + \int_{\varphi(\alpha)}^{\psi(\alpha)} \frac{\partial F}{\partial \alpha}(y,\alpha)\,\mathrm{d}y \tag{4}$$

(again, there is no need to prove this).

(a) Use the above formula for differentiating an integral depending on parameter to find $u_t(x,t)$ and $u_x(x,t)$ (where $u(x,t)$ is given by (3)).

(b) Use the above formula for differentiating an integral depending on parameter to find $u_{tt}(x,t)$ and $u_{xx}(x,t)$ and check that these derivatives satisfy the wave equation.

(c) Check that the expression for $u(x,t)$ given by the D'Alembert's formula satisfies the initial conditions $u(x,0) = 0$ and $u_t(x,0) = h(x)$.

Problem 4. [Compatibility between the PDE and the Neumann BC]

Consider a domain D in $\mathbb{R}^3$, and let ∂D be its boundary, with unit normal $\mathbf{n}$ pointing outwards. If $\mathbf{x} \in \partial D$ is a point on the boundary of D, let

$$\frac{\partial u}{\partial n}(\mathbf{x}) := D_{\mathbf{n}} u(\mathbf{x}) := \mathbf{n} \cdot \nabla u(\mathbf{x}) \ , \qquad \mathbf{x} \in \partial D \ ,$$

be the directional derivative of u in the direction of the outward unit normal vector $\mathbf{n}(\mathbf{x})$ at the point $\mathbf{x} \in \partial D$.

Consider the Neumann boundary value problem Poisson equation in D,

$$\begin{aligned} \Delta u = f \qquad & \text{in } D \ , \\ \left.\frac{\partial u}{\partial n}\right|_{\partial D} = g \qquad & \text{on } \partial D \ . \end{aligned} \tag{5}$$

Here $\Delta u = \nabla \cdot \nabla u = \operatorname{div} \operatorname{grad} u$ is the Laplacian of the scalar function $u : \mathbb{R}^3 \to \mathbb{R}$, and $f : D \to \mathbb{R}$ and $g : \partial D \to \mathbb{R}$ are given functions defined respectively in D and on ∂D.

(a) Use the Divergence Theorem to show that

$$\iiint_D f \, \mathrm{d}V = \oiint_{\partial D} g \, \mathrm{d}S \tag{6}$$

is a necessary condition for existence of a solution of the Neumann boundary value problem (5).

(b) Recall that the concentration $U(\mathbf{x}, t)$ of a certain substance in a homogeneous and non-moving medium satisfies the *diffusion equation*,

$$U_t(\mathbf{x}, t) = \alpha^2 \Delta U(\mathbf{x}, t) + \Psi(\mathbf{x}, t) \ , \tag{7}$$

where $\alpha^2 > 0$ is a positive constant (depending on the properties of the medium). Here $\Psi(\mathbf{x}, t)$ is the volume density of the rate of production of the substance at the point $\mathbf{x}$ and time t, i.e., the amount of substance produced in unit volume during one unit of time by some sources (if $\Psi(\mathbf{x}, t)$ is negative, this means that the substance was removed by a chemical reaction or in some other way).

If the sources of the substance are time-independent (i.e., $\Psi = \Psi(\mathbf{x}, t)$) and the boundary conditions also do not depend on the time, then the concentration $U(\mathbf{x}, t)$ will tend to some time-independent limit as $t \to \infty$): $u(\mathbf{x}) := \lim_{t \to \infty} U(\mathbf{x}, t)$. Since in this limit the time derivative in (7) vanishes, the time-independent concentration $u = u(\mathbf{x})$ will satisfy the Poisson equation

$$\Delta u(\mathbf{x}) = -\frac{1}{\alpha^2} \, \Psi(\mathbf{x})$$

(where we assume that the function $\Psi(\mathbf{x})$ is also time-independent) which gives a physical interpretation of the PDE in the boundary value problem (5).

Give a physical interpretation of the necessary condition (6) for the existence of a solution of the boundary value problem (5).

Hint: The meaning of the Neumann boundary condition is that the flux of the substance through the boundary ∂D is controlled, i.e., that we control the normal derivative $\frac{\partial u}{\partial n}(\mathbf{x}) = g(\mathbf{x})$ at each point $\mathbf{x}$ of the boundary ∂D. Recall that the diffusion obeys Fick's law, $\mathbf{j} = -\alpha^2 \nabla u$, where α^2 is a positive constant characterizing the medium, and $\mathbf{j}$ is the flux density of the flow of the substance. Think about the physical interpretation of the quantities $\oiint_{\partial D} \mathbf{j} \cdot \mathrm{d}\mathbf{S}$ and $\iiint_D \Psi \, \mathrm{d}V$.

Problem 5. [Application of Bessel functions]

The ODE

$$w''(z) + \frac{1}{z}\, w'(z) + \left(1 - \frac{n^2}{z^2}\right) w(z) = 0, \quad n = 0, 1, 2, 3, \dots \;, \tag{8}$$

is called the *Bessel differential equation.* Two linearly independent solutions of (8) are the functions $J_n(z)$ and $Y_n(z)$ called respectively *Bessel functions* and *Neumann functions* (or Bessel functions of first, resp., second kind), i.e., the general solution of (8) can be written as

$$w(z) = C_1 \, J_n(z) + C_2 \, Y_n(z) \; .$$

Consider the ODE

$$x f''(x) + f'(x) + f(x) = 0 \;, \quad x \geq 0 \; . \tag{9}$$

Change the independent variable from x to

$$s = 2\sqrt{x} \quad \text{or, equivalently,} \quad x = \frac{s^2}{4} \;,$$

and introduce the new function $S(s)$, by

$$S(s) := f\left(\frac{s^2}{4}\right) \quad \text{or, equivalently,} \quad f(x) = S(2\sqrt{x}) \; .$$

(a) Use the Chain Rule to express $f'(x)$ in terms of $S'(s)$. Note that the prime has different meaning in $f'(x)$ and $S'(s)$: $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x)$, while $S'(s) = \frac{\mathrm{d}S}{\mathrm{d}s}(x)$.

(b) Express $f''(x)$ in terms of the derivatives of $S(s)$.

(c) Substitute $f(x)$, $f'(x)$ and $f''(x)$ in the ODE (9) to rewrite it as and ODE for $S(s)$.

(d) Write the general solution of the ODE for $S(s)$ derived in part (c) as a linear combination $J_n(s)$ and $Y_n(s)$ for some n.

(e) Write down the general solution of the ODE (9).