MATH 4163 **Homework 3** **Due Thu, Sep 13, 2012**

Problem 1. In all parts of the problem below, you can use without deriving the following solutions of the heat equation $u_t(x,t) = \alpha^2 u_{xx}(x,t)$, $x \in [0, L]$, $t \geq 0$, with appropriate boundary conditions; the first expression is for zero temperature at both boundaries (homogeneous Dirichlet BCs, $u(0,t) = 0$, $u(L,t) = 0$), and the second is for zero heat flux at both boundaries (homogeneous Neumann BCs, $u_x(0,t) = 0$, $u_x(L,t) = 0$):

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left\{-\left(\frac{\alpha n\pi}{L}\right)^2 t\right\} \sin\frac{n\pi x}{L} \ ,$$

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \exp\left\{-\left(\frac{\alpha n\pi}{L}\right)^2 t\right\} \cos\frac{n\pi x}{L} \ .$$

(a) Solve the Dirichlet boundary value problem (BVP) below to find the temperature $u(x,t)$.

$$\begin{aligned} &u_t = 9u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\ &u(x,0) = 4\sin 2x + 7\sin 5x \ . \end{aligned}$$

In the expression for $u(x,t)$ take the limit $t \to \infty$ to find the asymptotic temperature, $u_\infty(x) := \lim_{t\to\infty} u(x,t)$. Explain why the expression you obtained for $u_\infty(x)$ is physically obvious.

(b) Use the hint below to solve the following Dirichlet BVP:

$$\begin{aligned} &u_t = u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\ &u(x,0) = 4\sin 4x \cos 2x \ . \end{aligned}$$

Hint: By using that $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$, one can derive the relations

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] \ , \quad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \ .$$

Similarly, from the identity $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$, one can derive an expression for the product $\sin\alpha\cos\beta$ in terms of sines and/or cosines of sums and differences of α and β:

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

(you do *not* need to write the derivation of this, but it is worth to think how one derives it). Use this expression to replace the product $4\sin 4x\cos 2x$ in the initial condition by a sum of trigonometric functions – this will make the problem very easy to solve (similar to what you did in part (a)).

(c) Solve the Neumann BVP below, find the asymptotic temperature, $u_\infty(x) := \lim_{t\to\infty} u(x,t)$, and explain why the expression you obtained for $u_\infty(x)$ is physically obvious.

$$
\begin{aligned}
&u_t = 9u_{xx} , \qquad x \in [0,5] , \quad t \geq 0 , \\
&u_x(0,t) = 0 , \quad u_x(5,t) = 0 , \\
&u(x,0) = 7 + 6\cos 2\pi x .
\end{aligned}
$$

(d) Solve the Neumann BVP below.

$$
\begin{aligned}
&u_t = 9u_{xx} , \qquad x \in [0,2] , \quad t \geq 0 , \\
&u_x(0,t) = 0 , \quad u_x(2,t) = 0 , \\
&u(x,0) = f(x) := \begin{cases} x & \text{for } x \in [0,1] , \\ 2 - x & \text{for } x \in [1,2] . \end{cases}
\end{aligned}
$$

You may use either the sine or the cosine Fourier expansion (you choose which one) of the function f from the initial condition, given below (you do *not* need to derive them):

$$
\begin{aligned}
f(x) &= \frac{8}{\pi^2}\left(\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \frac{1}{7^2}\sin\frac{7\pi x}{2} + \cdots\right) \\
&= 1 - \frac{16}{\pi^2}\left(\frac{1}{2^2}\cos\pi x + \frac{1}{6^2}\cos 3\pi x + \frac{1}{10^2}\cos 5\pi x + \frac{1}{14^2}\cos 7\pi x + \cdots\right) .
\end{aligned}
$$

Problem 2. Consider the problem about the stationary temperature distribution in the rectangle $x \in [0,a]$, $y \in [0,b]$ (which can be symbolically written as $(x,y) \in [0,a] \times [0,b]$) if there are no sources of heat in the rectangle, hence the temperature $u(x,y)$ satisfies Laplace's equation $\Delta u(x,y) = 0$. The temperature at the sides of the rectangle is maintained as follows:

$$
\begin{aligned}
&u(0,y) = 0 , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = \sin\frac{3\pi x}{a} , \qquad u(x,b) = 5\sin\frac{7\pi x}{a} \qquad \text{for } x \in [0,a] .
\end{aligned}
$$

(a) Solve the BVP

$$
\begin{aligned}
&\Delta u(x,y) = 0 , \qquad (x,y) \in [0,a] \times [0,b] \\
&u(0,y) = 0 , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = 0 , \qquad u(x,b) = 5\sin\frac{7\pi x}{a} \qquad \text{for } x \in [0,a] .
\end{aligned}
$$

Recall that one looks for a solution of this problem in the form

$$
u(x,y) = \sum_{n=1}^{\infty}\left(A_n \mathrm{e}^{\frac{n\pi y}{a}} + B_n \mathrm{e}^{-\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a}
$$

or, equivalently, in the form

$$u(x,y) = \sum_{n=1}^{\infty} \left(C_n \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a} \right) \sin \frac{n\pi x}{a} \ .$$

(b) Use the hint below to solve the BVP

$$\begin{aligned}
&\Delta u(x,y) = 0 \ , \qquad (x,y) \in [0,a] \times [0,b] \\
&u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = \sin \frac{3\pi x}{a} \ , \qquad u(x,b) = 0 \qquad \text{for } x \in [0,a] \ .
\end{aligned}$$

Hint: Let $Y_n(y)$ stand for the functions in the expansion

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y)\, X_n(x) \ ,$$

where because of the homogeneous boundary conditions at $x = 0$ and $x = a$ the functions $X_n(x)$ are given by $X_n(x) = \sin \frac{n\pi x}{a}$. Then the general solution of the ODE for $Y_n(y)$ is

$$Y_n(y) = C_n \cosh \frac{n\pi y}{a} + D_n \sinh \frac{n\pi y}{a} \ .$$

Show that the homogeneous boundary condition at $y = b$ implies that

$$\begin{aligned}
Y_n(y) &= E_n \left(\sinh \frac{n\pi b}{a} \cosh \frac{n\pi y}{a} - \cosh \frac{n\pi b}{a} \sinh \frac{n\pi y}{a} \right) \\
&= E_n \sinh \frac{n\pi (b-y)}{a}
\end{aligned}$$

(where E_n are constants arbitrary at the moment); here we have used the fact that hyperbolic sine satisfies

$$\sinh(\alpha \pm \beta) = \sinh \alpha \cosh \beta \pm \cosh \alpha \sinh \beta \ .$$

Now impose the remaining boundary condition to find the constants E_n (of which only one will be non-zero).

(c) Since the equation $\Delta u = 0$ is linear and homogeneous (i.e., with a zero right-hand side), the principle of superposition holds similarly to the case of ordinary differential equations. Using this fact, use your results from parts (a) and (b) to write down the solution of the BVP

$$\begin{aligned}
&\Delta u(x,y) = 0 \ , \qquad (x,y) \in [0,a] \times [0,b] \\
&u(0,y) = 0 \ , \qquad u(a,y) = 0 \qquad \text{for } y \in [0,b] \\
&u(x,0) = \sin \frac{3\pi x}{a} \ , \qquad u(x,b) = 5 \sin \frac{7\pi x}{a} \qquad \text{for } x \in [0,a] \ .
\end{aligned}$$

Problem 3. From the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\alpha^2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

for the function $u(r, t)$ (where α is a positive constant), derive the ordinary differential equations that are implied by the method of separation of variables. In other words, set $u(r, t) = R(r)\,T(t)$ and derive ordinary differential equations for the functions R and T; do not forget that there should be a constant coming from the separation of variables (we denoted it μ in class). Do *not* attempt to solve the equations you obtain.