MATH 4073 Homework 3 Due Wed, 09/21/16

Problem 1. [Calculating trillions of digits of π]

On August 2, 2010, Shigeru Kondo used Alexander Yee announced that they have calculated 5,000,000,000,000 digits of π; on December 28, 2013, they improved their own record by computing 12.1 trillion digits of π. They used the a program called y-cruncher, developed by Yee, and performed their computations on a single desktop computer built by Kondo; the computation of 12.1 trillion digits took about 94 days You can see their announcement and details on their work at

`http://http://www.numberworld.org/misc_runs/pi-12t/`

`http://www.numberworld.org/misc_runs/pi-5t/details.html`

In their computations Kondo and Yee used the following formula derived by the brothers David and Gregory Chudnovsky, who relied on some ideas of the famous Inidian mathematician Srinivasa Ramanujan (1887–1920):

$$\frac{1}{\pi} = \frac{\sqrt{10005}}{4270934400} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(k!)^3\,(3k)!} \, \frac{13591409 + 545140134k}{640320^{3k}} \,.$$

In this problem you will use Mathematica to find the rate of convergence of the right-hand side of this formula to the exact value of $\frac{1}{\pi}$. You can define the function `chud[n]` which computes the sum of the first `n` terms of Chudnovsky's formula:

```
termPi[k_]=(-1)^k*(6*k)!/(k!)^3/(3*k)!*(13591409+545140134*k)/640320^(3*k)

chud[n_]=Sqrt[10005]/4270934400*Sum[termPi[k], {k, 0, n}]
```

After you type each line in Mathematica, press SHIFT, hold it down, and press RETURN. The underscores after `k` and `n` in `termPi[k_]` and `chud[n_]` tell Mathematica that we are defining new functions, and `k` and `n` the variables of these functions.

To find the numerical value with accuracy of 1000 digits of the difference between the exact value of $\dfrac{1}{\pi}$ and the partial sum of the sum containing, say, 8 terms – which in our notations will be equal to `chud[7]` – you can type the following:

```
N[chud[7] - 1/Pi, 1000]
```

There will a problem, however, and Mathematica will complain that its internal precision limit is not enough for the computation (try it!). That is why you have to type

```
Block[{$MaxExtraPrecision = 1000}, N[chud[7] - 1/Pi, 1000]]
```

(a) Compute the numerical values of the absolute error $E_n = \left|\frac{1}{\pi} - \texttt{chud}[n]\right|$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$, and write your results in a table (there is no need to write more than 3–4 digits of accuracy of E_n in the table).

(b) For the values of n used in part (a), show that your numerical results give $\frac{E_{n+1}}{E_n} \approx 10^{-14}$. Can you express E_n approximately in terms of E_0? I do not want anything sophisticated, just a VERY ROUGH approximate formula.

Problem 2. [Theoretical computations of α and λ]

Directly from the definition, find the rates of convergence α and the asymptotic error constants λ for each of the sequences (all of which tend to 0)

$$\text{(a)}\quad x_n = \frac{1}{n^2}\ ; \qquad\qquad \text{(b)}\quad x_n = 7^{-n}\ ; \qquad\qquad \text{(c)}\quad x_n = 10^{-5^n}\ .$$

Problem 3. [Empirical computations of α and λ]

The concepts of asymptotic error constant λ and especially order of convergence α are very important when one is using an *iterative method*, i.e., a method in which the exact solution of the problem is found as a limit of a sequence of approximate values. If the exact value p is a limit of a sequence $\{p_n\}_{n=0}^{\infty}$ of approximate values, then the *error* at the nth step of the iteration is $E_n := |p_n - p|$. The rate of decreasing of E_n is one of the most important characteristics of an iterative method.

Assume that the sequence $(p_n)_{n=0}^{\infty}$ is generated by some iterative method for finding a root of an equation. Also assume that we know that the sequence $(p_n)_{n=0}^{\infty}$ converges to some number p of some order α with some asymptotic error constant λ, but we don't know the values of α and λ. The goal of this problem is to develop a method for determining the numerical value of α from the numerical values of the members of the sequence $(p_n)_{n=0}^{\infty}$. Let $\ell_n := \log_{10} E_n$.

(a) Show that for large n, the following approximate identity holds:

$$\ell_n - \alpha \ell_{n-1} \approx \log_{10} \lambda\ .$$

Hint: Just look at the definition of order of convergence.

(b) Using the approximate identity derived in (a) show that

$$\alpha \approx \frac{\ell_n - \ell_{n+1}}{\ell_{n-1} - \ell_n}\ .$$

Note that this approximate formula for α does not depend on the base of the logarithms; if ℓ_n is defined as the log base 10 of E_n, the formula will remain the same.

(c) The data in Table 1 come from applying the *Newton method* and the *secant method* to find the root of the equation

$$x + \sin x = 1\ ,$$

whose exact value is $p = 0.51097342938856910952001397114508063204535889262\ldots$. Use the formula derived in part (b) to find empirically the order of convergence α for these two methods.

Problem 4. [Failure of the secant method]

Let $f(x) = 3 - x^2$. You can easily check that this equation has a root in the interval $[-2, 1]$ (in fact, you can easily find the root exactly); you do *not* need to do this here.

Now pretend that you cannot solve the equation $f(x) = 0$ explicitly, and apply the secant method to try to find a root of this equation starting from the values $p_0 = -2$ and $p_1 = 1$. Perform

n	ℓ_n, Newton	ℓ_n, secant
0	-0.31067	-0.31067
1	-2.85988	-1.49389
2	-7.84087	-2.54052
3	-17.7179	-4.90935
4	-37.4715	-8.33484
5	-76.9787	-14.1282
6	-155.993	-23.3471
7	-314.022	-38.3595
8	-630.079	-62.5907
9	-1262.19	-101.834
10	-2526.42	-165.309
11	-5054.88	-268.027
12	-10111.8	-434.220

Table 1: Log_{10} of the errors of the Newton and the secant methods.

several steps of the secant method "by hand" (i.e., do not use a computer, but write clearly your calculations). What do you observe? Draw a picture and explain why the secant method failed so miserably.

Problem 5. [Failure of the Newton's method]

Let the function f be defined as

$$f(x) = 5x - x^3 .$$

(a) Find the solutions of the equation $f(x) = 0$ exactly (by hand – it is very easy!).

(b) Write an explicit formula (for this particular choice of $f(x)$) for an iterative procedure based on Newton's method.

(c) Compute by hand the first several iterates of the Newton iteration starting from $p_0 = 1$ and without any roundoff error. Does the iterative procedure converge?

(d) Draw the graph of the function f and carefully give a graphical and verbal explanation of your observations in (c).

(e) If $p_0 = 1$ or $p_0 = -1$, Newton's method will be in trouble, as you have already discovered.

Think of other value(s) of p_0 for which Newton's method will fail for the function $f(x)$ from this problem. Explain your reasoning.

Hint: Besides 1 and -1, there are two more values of p_0 that will give trouble to Newton right away.