MATH 3423 **Homework 3** **Due Mon, 09/19/16**

Problem 1.

(a) Directly from the definition of a contour integral in $\mathbb{C}$, $\displaystyle\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t$ (where $z(t)$, $t \in [a,b]$ is a parameterization of the contour C), calculate the integral $\displaystyle\int_C z\,\mathrm{d}z$, where the contour C is given by the parametric equation $z(t) = 1 + \mathrm{e}^{\mathrm{i}t}$, and the parameter t varies from 0 to $\frac{\pi}{4}$.

(b) A much faster way to compute the integral from part (a) is to use the Theorem on page 888 of the book. Compute the value of the integral, and compare your result with the result from part (a). Did you use the specific form of the contour C in your calculation in this part of the problem?

(c) Evaluate the value of the integral $\displaystyle\int_C \left(z^6 + z\mathrm{e}^{z^2}\right)\mathrm{d}z$, where the contour C is a segment of a straight line from starting at 2 and ending at i.

Problem 2. Evaluate $\displaystyle\oint_C \frac{\mathrm{d}z}{(z-a)^5}$, where C is a simple closed contour of arbitrary shape (traversed in positive direction), encircling the point $a \in \mathbb{C}$.

Problem 3. Use Cauchy's Integral Formula to evaluate $\displaystyle\oint_C \frac{\tan z}{4z - \pi}\,\mathrm{d}z$, where C is the contour (traversed in positive direction) given by

(a) $|z| = 1$;

(b) $|z| = \frac{1}{2}$.

Problem 4. Evaluate $\displaystyle\oint_C \frac{\mathrm{e}^{\mathrm{i}z}}{z^2+1}\,\mathrm{d}z$, where C is the contour given by $|z| = 2$ (traversed in positive direction).

Problem 5. Evaluate $\displaystyle\oint_C \frac{\cos z}{z(z-2i)^2}\,\mathrm{d}z$, where C is the circle $|z - 3i| = 2$ (traversed in positive direction).

Hint: See Example 4 on page 897. Be careful about which singularities lie inside C.

Problem 6.

(a) Use the formula for the sum of a geometric series, $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, $z \in \mathbb{C}$, $|z| < 1$, to show that the sum of the series $\sum_{n=0}^{\infty} r^n \mathrm{e}^{\mathrm{i}n\theta}$ (where $r \geq 0$ and θ are real numbers) is

$$\sum_{n=0}^{\infty} r^n \mathrm{e}^{\mathrm{i}n\theta} = \frac{1 - r\cos\theta + \mathrm{i}\, r\sin\theta}{1 - 2r\cos\theta + r^2} \; .$$

For what values of r and θ is your result valid?

(b) Use the formula derived in part (a) to prove that $\sum_{n=0}^{\infty} r^n \cos(n\theta) = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$. For what values of r and θ is this formula valid?

(c) Use the formula for the sum of a geometric series to show that the series $\sum_{n=1}^{\infty} n r^{n-1} \mathrm{e}^{\mathrm{i}(n-1)\theta}$ (where again $r \geq 0$ and θ are real numbers) converges when $r \in [0, 1)$, and its sum is

$$\sum_{n=1}^{\infty} n r^{n-1} \mathrm{e}^{\mathrm{i}(n-1)\theta} = \frac{1}{(1 - r\mathrm{e}^{\mathrm{i}\theta})^2} \; .$$

Hint: How can you get nz^{n-1} from z^n?

Problem 7. Determine the radius and the disk of convergence of the power series $\sum_{n=0}^{\infty} \frac{(z-i)^n}{n^2}$.

Problem 8. Consider the function $f(z) = z^3 \cosh \frac{1}{z}$.

(a) Use the Taylor series $\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ to derive the Taylor series of $\cosh z$.

(b) Find the Laurent expansion of $f(z)$ about the point $a = 0$.

(c) The point $a = 0$ is a singularity of $f(z)$. What kind of singularity?

Problem 9. Consider the function $f(z) = \frac{1}{z^3(z-5)}$.

(a) Find and classify the singularities of $f(z)$.

(b) Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $0 < |z| < 5$.

(c) Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $5 < |z|$.

(d) Compute the residue of $f(z)$ at $a = 0$. If you use some of the previously obtained results to answer this question, please specify clearly what you used.

Problem 10. Find *all* Laurent expansions of $f(z) = \frac{1}{z-2}$ about the point $a = i$.