

Problem 1. The solution of the initial value problem for the wave equation

$$\begin{aligned}\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) &= 0, \\ u_t(x, 0) &= h(x)\end{aligned}$$

(where v is a positive constant) is given by a particular case of the so-called *D'Alembert's formula*,

$$u(x, t) = \frac{1}{2v} \int_{x-vt}^{x+vt} h(z) \, dz \quad (1)$$

(you do not need to prove this).

The following formula holds for differentiating an integral whose limits and integrand depend on some parameter α :

$$\frac{d}{d\alpha} \int_{\phi(\alpha)}^{\psi(\alpha)} F(y, \alpha) \, dy = F(\psi(\alpha), \alpha) \psi'(\alpha) - F(\phi(\alpha), \alpha) \phi'(\alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial F}{\partial \alpha}(y, \alpha) \, dy \quad (2)$$

(again, there is no need to prove this).

- Use the above formula for differentiating an integral depending on parameter to find $u_t(x, t)$ and $u_x(x, t)$ (where $u(x, t)$ is given by (1)).
- Use the above formula for differentiating an integral depending on parameter to find $u_{tt}(x, t)$ and $u_{xx}(x, t)$ and check that these derivatives satisfy the wave equation.
- Check that the expression for $u(x, t)$ given by the D'Alembert's formula satisfies the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = h(x)$.

Problem 2. Recall the fact that $\int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\pi}$. By change of variables in the integral, one can easily obtain that

$$\int_{-\infty}^{\infty} e^{-\alpha y^2} \, dy = \sqrt{\frac{\pi}{\alpha}}. \quad (3)$$

Below you will obtain some facts about integrals of the form $\int_{-\infty}^{\infty} y^n e^{-y^2} \, dy$, for some $n \in \mathbb{N}$.

- Explain why $\int_{-\infty}^{\infty} y^{2n-1} e^{-y^2} \, dy = 0$ for any $n \in \mathbb{N}$.

Hint: The reason is *very* simple!

(b) Use formula (2) from Problem 1 to differentiate both parts of equation (3) with respect to the parameter α , and then set $\alpha = 1$, to obtain an expression for $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy$.

(c) Similarly to part (b), find $\int_{-\infty}^{\infty} y^4 e^{-y^2} dy$.

Problem 3. Find the four 4th roots of i .

Problem 4. Starting with $\int_0^{\infty} e^{-\beta^2 y^2} dy = \frac{\sqrt{\pi}}{2\beta}$, let $\beta = \frac{1-i}{\sqrt{2}}$ (and notice that $\beta^2 = -i$) to show that $\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \sqrt{\frac{\pi}{8}}$.

Problem 5. Directly from the definition of $\sinh z$, show that the Taylor expansion of $\sinh z$ around 0 is $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$.

Problem 6. Determine $\ln(-i)$ and $\text{Ln}(-i)$.

Problem 7.

(a) Show that $\arcsin z = -i \ln(iz \pm \sqrt{1-z^2})$.

Hint: Solve $\sin w = z$ (i.e., $\frac{e^{iw} - e^{-iw}}{2i} = z$) for z . You can set $\xi := e^{iw}$, and rewrite $\frac{e^{iw} - e^{-iw}}{2i} = z$ as a quadratic equation for ξ .

(b) Use your result from part (a) to solve the equation $\sin w = 2$. Note that this equation has no solution if w is real.

(c) Directly from the definition of the sine function, show that $\sin\left(\frac{\pi}{2} - i \ln(2 \pm \sqrt{3})\right) = 2$.

Problem 8. Evaluate $\sqrt{-1}^{\sqrt{-1}}$ (i.e., i^i).

Problem 9. Show that $\lim_{z \rightarrow 0} \frac{z^*}{z}$ does not exist by taking the limit along the ray $y = mx$, where m is a real constant.

Problem 10. Classify all singularities of $f(z) = \frac{z}{(z^2 + 4)^2}$.

Problem 11. Let $u(x, y) = x^3 - 3xy^2$. Use Cauchy-Riemann equations to find a function $v(x, y)$ such that the function $f(z) = u(x, y) + i v(x, y)$ is differentiable (where $z = x + iy$).