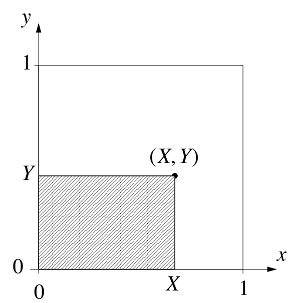
MATH 5763   Homework 2   Due Thu, 1/31/2019

Problem 1. Let X and Y be independent random variables, each with distribution Uniform$(0,1)$. Then the point with coordinates (X,Y) is a random vector that is uniformly distributed in the unit square, i.e., its probability density function is

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \ , \quad 0 \le y \le 1 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Let A be the random variable equal to the area of the rectangle with vortices at the points $(0,0)$, $(0,Y)$, $(X,0)$, and (X,Y) – see the figure. Clearly, A is a continuous random variable taking values in the interval $[0,1]$.



(a) In the (x,y) plane, sketch the domain determined by the inequalities $0 \le x \le 1$, $0 \le y \le 1$, $xy \le a$ (where a is a value between 0 and 1).

(b) Show that the cumulative distribution function of A,

$$F_A(a) = \mathbb{P}(A \le a) = \mathbb{P}(XY \le a) = \iint_{xy \le a} f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y \ ,$$

is given by

$$F_A(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0] \ , \\ a\,(1 - \ln a) & \text{if } a \in (0,1] \ , \\ 1 & \text{if } a \in [1, \infty) \ . \end{cases}$$

Hint: What you drew in part (a) will be useful.

(c) Find the probability density function, $f_A(a)$, of A. Be sure to specify $f_A(a)$ for all values of a.

(d) Determine the expected value $\mathbb{E}[A]$ of the area A of the random square with sides of length X and Y.

Problem 2. The joint p.m.f. $p_{X,Y}(x_k, y_m) = \mathbb{P}(X = x_k, Y = y_m)$ of the discrete RVs X and Y has values given in the table.

	$Y=1$	$Y=3$	$Y=4$
$X=5$	0	$\frac{1}{15}$	$\frac{2}{15}$
$X=7$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(a) Find the marginal p.m.f.s $p_X(x_k)$ and $p_Y(y_m)$.

(b) Find the expected value $\mathbb{E}[X]$ of X.

(c) Compute the conditional p.m.f.s $p_{X|Y}(x_k|y_m) = \mathbb{P}(X = x_k|Y = y_m)$ for $y_m = 1, 3, 4$.

(d) Find the conditional expectations $\mathbb{E}[X|Y = y_m]$ for $y_m = 1, 3, 4$.

(e) The quantity $\mathbb{E}[X|Y]$ can be considered as a random variable which is a function of Y, and written as a linear combination of the indicator functions of the sets $\{Y = y_m\} = Y^{-1}(y_m)$:

$$\mathbb{E}[X|Y] = \sum_m \mathbb{E}[X|Y = y_m]\, I_{Y^{-1}(y_m)} \ .$$

Use your result from part (d) to write $\mathbb{E}[X|Y]$ in this form (using concrete numbers).

(f) Use the representation of $\mathbb{E}[X|Y]$ from part (e) to find its expectation, $\mathbb{E}\left[\mathbb{E}[X|Y]\right]$.

Hint: Recall the linearity property of the expectation, and use that the expectation of the indication function of any $A \in \mathcal{F}$ is $\mathbb{E}(I_A) = 0 \cdot \mathbb{P}(A^c) + 1 \cdot \mathbb{P}(A) = \mathbb{P}(A)$. According to the so-called *tower rule*, $\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X]$, you should obtain in a different way the same value as in part (b).

Problem 3. A frog lays Y eggs, where $Y \sim \mathrm{Poi}(\lambda)$ is a Poisson RV with parameter $\lambda > 0$, i.e., the p.m.f. of Y is $p_Y(n) = \mathrm{e}^{-\lambda}\,\frac{\lambda^n}{n!}$, $n = 0, 1, 2, \ldots$, where $0! := 1$.

Each egg survives independently of the survival of the other eggs, with probability $p \in (0, 1)$. From this one can derive that the number X of surviving eggs is a binomial RV with parameters Y and p, which can be written symbolically as $X \sim \mathrm{Bin}(Y, p)$. This means that the RV X, conditioned on the event $\{Y = n\}$, is binomial with parameters n and p: $X \,|\, \{Y = n\} \sim \mathrm{Bin}(n, p)$, i.e., that the conditional p.m.f. of X conditioned on the value of Y is

$$p_{X|Y}(k|n) = \binom{n}{k} p^k\,(1-p)^{n-k} \ , \qquad k = 0, 1, \ldots, n \ ,$$

where $\displaystyle\binom{n}{k} := \frac{n!}{k!\,(n-k)!}$ are the binomial coefficients.

Show by a direct calculation of the p.m.f. of X that $X \sim \mathrm{Poi}(\lambda p)$.

Hint: Use equation (3) derived in Food For Thought Problem 1 below. You may also find some of the following facts useful:

- series expansion of the exponential function: $\displaystyle \mathrm{e}^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \ , \qquad t \in \mathbb{R};$

- binomial formula: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\, y^{n-k}$ for $x, y \in \mathbb{R}$, $n \in \{0, 1, 2, 3, \ldots\}$;
- geometric series: $\frac{1}{1-q} = \sum_{k=0}^{\infty} q^k$ whenever $|q| < 1$.

Problem 4. Let X be a stationary discrete-time discrete-state space Markov chain with state space S consisting of two states, 0 and 1, and let the 1-step transition probability matrix of the stochastic process be

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & 0 \end{bmatrix} , \tag{1}$$

where $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$. Let

$$\rho_{ij}^{(n)} := \mathbb{P}\left(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j | X_0 = i\right) \ , \qquad n \geq 1 \ , \quad i, j \in \{0, 1\}$$

be the probability of moving to state j, from the initial state i, *for the first time* at the nth transition. Recall the relations

$$p_{ij}^{(n)} = \sum_{k=1}^{n} \rho_{ij}^{(k)}\, p_{jj}^{(n-k)} \ , \tag{2}$$

where $p_{ij}^{(n)}$ are the entries of the n-step transition probability matrix $\mathbf{P}^{(n)}$. Assume (quite reasonably) that $\mathbf{P}^{(0)} = \mathbf{I}$ (the identity matrix).

(a) Compute explicitly $\mathbf{P}^{(0)}$, $\mathbf{P}^{(1)}$, $\mathbf{P}^{(2)}$, and $\mathbf{P}^{(3)}$.

(b) Use (2) to compute the value of $\rho_{00}^{(1)}$.

(c) Use (2) and the value of $\rho_{00}^{(1)}$ (found in (b)) to compute the value of $\rho_{00}^{(2)}$.

(d) Use (2) and the values of $\rho_{00}^{(1)}$ and $\rho_{00}^{(2)}$ (found in (b) and (c)) to compute the value of $\rho_{00}^{(3)}$.

Food for Thought Problem 1. Let X and Y be discrete RVs taking values x_k and y_m, where the indices k and m run over discrete sets (finite or infinite).

Use the notation $\{Y = y_m\} = Y^{-1}(y_m) = \{\omega \in \Omega \,|\, Y(\omega) = y_m\}$ and the fact that the events $\{Y = y_m\}$ form a partition of the sample space Ω, to convince yourselves that the following derivation of the p.m.f. of the RV X through conditioning on the RV Y is correct:

$$\begin{aligned} p_X(x_k) &= \mathbb{P}(X = x_k) = \mathbb{P}(\{X = x_k\} \cap \Omega) = \mathbb{P}\Big(\{X = x_k\} \cap \bigcup_m \{Y = y_m\}\Big) \\ &= \mathbb{P}\Big(\bigcup_m \{X = x_k, Y = y_m\}\Big) = \sum_m \mathbb{P}\left(X = x_k, Y = y_m\right) \\ &= \sum_m \mathbb{P}\left(X = x_k \,|\, Y = y_m\right)\, \mathbb{P}(Y = y_m) = \sum_m p_{X|Y}(x_k | y_m)\, p_Y(y_m) \ . \end{aligned} \tag{3}$$

Food for Thought Problem 2. Prove the recursive relation (2).

Solution: Let $i \in \{0,1\}$, $j \in \{0,1\}$, and $n \geq 1$ be fixed. Notice that the events

$$A_k := \{ \text{ the MC reaches state } j \text{ for the first time in exactly } k \text{ steps } \} \ .$$

Clearly, for a given n, the time k when the MC reaches state j for the first time (starting from state i) can take values 1, 2, ..., n. It is obvious that the events A_k form a partition of the sample space Ω (why?):

$$\bigcup_{k=1}^{n} A_k = \Omega \ , \qquad A_k \cap A_{k'} = \varnothing \ \text{ for } k \neq k' \ . \tag{4}$$

Note that

$$\rho_{ij}^{(k)} = \mathbb{P}(A_k \,|\, X_0 = i) \ . \tag{5}$$

Using (4) and (5), we obtain

$$\begin{aligned}
p_{ij}^{(n)} &= \mathbb{P}\big(X_n = j \,\big|\, X_0 = i\big) \\
&= \mathbb{P}\big(\{X_n = j\} \cap \Omega \,\big|\, X_0 = i\big) \\
&= \mathbb{P}\left(\{X_n = j\} \cap \bigcup_{k=1}^{n} A_k \,\middle|\, X_0 = i\right) \\
&= \mathbb{P}\left(\bigcup_{k=1}^{n} \left(\{X_n = j\} \cap A_k\right) \,\middle|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(\{X_n = j\} \cap A_k \,\big|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \frac{\mathbb{P}\left(\{X_n = j\} \cap A_k \cap \{X_0 = i\}\right)}{\mathbb{P}(X_0 = i)} \\
&= \sum_{k=1}^{n} \frac{\mathbb{P}\left(\{X_n = j\} \cap A_k \cap \{X_0 = i\}\right)}{\mathbb{P}\left(A_k \cap \{X_0 = i\}\right)} \ \frac{\mathbb{P}\left(A_k \cap \{X_0 = i\}\right)}{\mathbb{P}(X_0 = i)} \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_n = j \,|\, A_k \cap \{X_0 = i\}\right) \ \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_n = j \,|\, A_k\right) \ \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_n = j \,|\, X_k = j\right) \ \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_{n-k} = j \,|\, X_0 = j\right) \ \mathbb{P}\left(A_k \,|\, X_0 = i\right) \\
&= \sum_{k=1}^{n} p_{jj}^{(n-k)} \, \rho_{ij}^{(k)} \ .
\end{aligned}$$