

Problem 1. Let X and Y be independent Poisson random variables with respective parameters λ and μ . Show that:

- (a) $X + Y$ is Poisson with parameter $\lambda + \mu$;
- (b) the conditional distribution of X given that $X + Y = n$ is binomial, and find its parameters.

Problem 2. Each night different meteorologists give us the probability that it will rain the next day. To judge how well they predict, we will give a score to each of them as follows. If a meteorologist says that it will rain with probability p , then he or she will receive a score of

$$\begin{aligned} 1 - (1 - p)^2 & \text{ if it rains ,} \\ 1 - p^2 & \text{ if it does not rain .} \end{aligned}$$

We will keep track of scores over a certain long time period, and will conclude that a meteorologist with the highest average score is the best predictor of weather.

Suppose now that a given meteorologist is aware of this and wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^* , what value of p should he or she report so as to maximize the expected score?

Problem 3. If X , Y , and Z are random variables, prove the following properties of conditional expectation:

- (a) (linearity) $\mathbb{E}[aY + bZ|X] = a\mathbb{E}[Y|X] + b\mathbb{E}[Z|X]$ for $a, b \in \mathbb{R}$;
- (b) $\mathbb{E}[Y|X] \geq 0$ if $Y \geq 0$ (i.e., if the random variable Y takes only non-negative values with non-zero probability);
- (c) if X and Y are independent, then $\mathbb{E}[Y|X] = \mathbb{E}[Y]$;
- (d) (the ‘pull-through property’) $\mathbb{E}[Yg(X)|X] = g(X)\mathbb{E}[Y|X]$ for any suitable function $g : \mathbb{R} \rightarrow \mathbb{R}$ (‘suitable’ here simply mean ‘such that the expressions written make sense’).

Problem 4. The table below displays most of the values of the (joint) probability mass function $p_{X,Y}(x, y)$ of the random vector (X, Y) , where X takes values 1, 2, and 3, while Y takes values 4 and 5.

- (a) What is the value of the missing entry? Why?
- (b) Find the marginal p.m.f.s $p_X(x)$ and $p_Y(y)$.

	1	2	3
4	0.05	?	0.15
5	0.00	0.40	0.30

- (c) Find the expectations $\mathbb{E}X$ and $\mathbb{E}Y$.
- (d) Find the conditional expectation $\mathbb{E}[Y|X]$.
- (e) Using your results from part (d), compute $\mathbb{E}[\mathbb{E}[Y|X]]$. Does the number that you obtained match your expected? Why?

Problem 5. Let X_1, X_2, \dots be identically distributed (but not necessarily independent) random variables with mean μ , and let N be a random variable taking values in the non-negative integers and independent of the X_i . Let $Y = X_1 + X_2 + \dots + X_N$. Show that $\mathbb{E}[Y|N] = \mu N$, and deduce that $\mathbb{E}[Y] = \mu \mathbb{E}[N]$.

Problem 6. The continuous random variables X and Y have joint p.d.f.

$$f_{X,Y}(x, y) = \begin{cases} 3 & \text{if } 0 \leq y \leq x^2 \leq 1 \text{ and } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(In other words, X and Y form a random vector with p.d.f. $f_{X,Y}(x, y)$ given by the expression above.)

- (a) In the (x, y) plane draw the area in which $f_{X,Y}(x, y)$ is nonzero.
- (b) Find the expectation of X^3 by using $f_{X,Y}(x, y)$ (without computing the marginal p.d.f. $f_X(x)$ first).
- (c) Compute the marginal p.d.f. $f_X(x)$.
- (d) Determine the marginal p.d.f. $f_Y(y)$.
- (e) Find the expectation of X^3 by using $f_X(x)$. You should obtain the same answer as in part (b).
- (f) Compute the conditional p.d.f. $f_{X|Y}(x|y)$.
Hint: By definition, $f_{X|Y}(x|y)$ is equal to 0 whenever $f_Y(y)$ is zero.
- (g) What is the conditional p.d.f. $f_{Y|X}(y|x)$?
- (h) Determine the conditional expectation $\mathbb{E}[X^3|Y = y]$.
- (i) Find the expectation of X^3 by using the tower rule and the expression for $\mathbb{E}[X^3|Y = y]$ obtained in part (h). The result must be the same as in parts (b) and (e).