MATH 5763 Homework 2 Due Fri, 2/4/11

Problem 1. Let X and Y be independent Poisson random variables with respective parameters λ and μ. Show that:

(a) $X + Y$ is Poisson with parameter $\lambda + \mu$;

(b) the conditional distribution of X given that $X + Y = n$ is binomial, and find its parameters.

Problem 2. Each night different meteorologists give us the probability that it will rein the next day. To judge how well they predict, we will give a score to each of them as follows. If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\begin{array}{ll} 1-(1-p)^2 & \text{if it rains ,} \\ 1-p^2 & \text{if it does not rain .} \end{array}$$

We will keep track of scores over a certain long time period, and will conclude that a meteorologist with the highest average score is the best predictor of weather.

Suppose now that a given meteorologist is aware of this and wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^*, what value of p should he or she report so as to maximize the expected score?

Problem 3. If X, Y, and Z are random variables, prove the following properties of conditional expectation:

(a) (linearity) $\mathbb{E}[aY + bZ|X] = a\mathbb{E}[Y|X] + b\mathbb{E}[Z|X]$ for $a, b \in \mathbb{R}$;

(b) $\mathbb{E}[Y|X] \geq 0$ if $Y \geq 0$ (i.e., if the random variable Y takes only non-negative values with non-zero probability);

(c) if X and Y are independent, then $\mathbb{E}[Y|X] = \mathbb{E}[Y]$;

(d) (the 'pull-through property') $\mathbb{E}[Yg(X)|X] = g(X)\mathbb{E}[Y|X]$ for any suitable function $g : \mathbb{R} \to \mathbb{R}$ ('suitable' here simply mean 'such that the expressions written make sense).

Problem 4. The table below displays most of the values of the (joint) probability mass function $p_{X,Y}(x, y)$ of the random vector (X, Y), where X takes values 1, 2, and 3, while Y takes values 4 and 5.

(a) What is the value of the missing entry? Why?

(b) Find the marginal p.m.f.s $p_X(x)$ and $p_Y(y)$.

	1	2	3
4	0.05	?	0.15
5	0.00	0.40	0.30

(c) Find the expectations $\mathbb{E}X$ and $\mathbb{E}Y$.

(d) Find the conditional expectation $\mathbb{E}[Y|X]$.

(e) Using your results from part (d), compute $\mathbb{E}[\mathbb{E}[Y|X]]$. Does the number that you obtained match your expected? Why?

Problem 5. Let X_1, X_2, ... be identically distributed (but not necessarily independent) random variables with mean μ, and let N be a random variable taking values in the non-negative integers and independent of the X_i. Let $Y = X_1 + X_2 + \cdots + X_N$. Show that $\mathbb{E}[Y|N] = \mu N$, and deduce that $\mathbb{E}[Y] = \mu\,\mathbb{E}[N]$.

Problem 6. The continuous random variables X and Y have joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} 3 & \text{if } \ 0 \le y \le x^2 \le 1 \text{ and } \ x \ge 0 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

(In other words, X and Y form a random vector with p.d.f. $f_{X,Y}(x,y)$ given by the expression above.)

(a) In the (x,y) plane draw the area in which $f_{X,Y}(x,y)$ is nonzero.

(b) Find the expectation of X^3 by using $f_{X,Y}(x,y)$ (without computing the marginal p.d.f. $f_X(x)$ first).

(c) Compute the marginal p.d.f. $f_X(x)$.

(d) Determine the marginal p.d.f. $f_Y(y)$.

(e) Find the expectation of X^3 by using $f_X(x)$. You should obtain the same answer as in part (b).

(f) Compute the conditional p.d.f. $f_{X|Y}(x|y)$.

Hint: By definition, $f_{X|Y}(x|y)$ is equal to 0 whenever $f_Y(y)$ is zero.

(g) What is the conditional p.d.f. $f_{Y|X}(y|x)$?

(h) Determine the conditional expectation $\mathbb{E}[X^3|Y = y]$.

(i) Find the expectation of X^3 by using the tower rule and the expression for $\mathbb{E}[X^3|Y = y]$ obtained in part (h). The result must be the same as in parts (b) and (e).