MATH 5763 **Homework 2** **Due Wed, 9/9/2015**

Problem 1. Let A and B be events with $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.4$. Find the conditional probability $\mathbb{P}(A|B)$ in the following cases:

(a) A and B are mutually exclusive (i.e., disjoint);

(b) $\mathbb{P}(A \cap B) = 0.1$;

(c) A implies B (i.e., every time A occurs, B also occurs).

Problem 2. Let X be a continuous RV uniformly distributed over the interval $[0, 1]$, i.e., the p.d.f. of X is

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Let X_1 and X_2 be independent continuous RVs modeled after the RV X, i.e., $f_{X_1}(x) = f_{X_2}(x) = f_X(x)$ for any $x \in \mathbb{R}$.

(a) Derive the expression for the c.d.f. $F_X(x)$ of the random variable X.

(b) Prove that the minimum, $X_{\min} = \min\{X_1, X_2\}$, of the RVs X_1 and X_2 has c.d.f.

$$F_{X_{\min}}(x) = 1 - [1 - F_X(x)]^2 \ .$$

Please point out at which point in your proof you used the independence of X_1 and X_2.

(c) Use your result for the c.d.f. of $X_{\min}$ from part (b) to find the p.d.f. $f_{X_{\min}}(x)$ of $X_{\min}$.

(d) What can you say about the expectation of $X_{\min}$ *without doing any calculations*? (Not the exact value, just say *something reasonable*, and explain how you came to this conclusion.)

(e) Now compute the exact value of $\mathbb{E}[X_{\min}]$.

Problem 3. Let X be a continuous RV uniformly distributed over the interval $[0, 1]$ as in Problem 1. Define the continuous RV Y to be a function of the RV X defined as

$$Y = -\ln X \ .$$

(a) Find the interval of values where Y takes values. (Do not worry that X can take value 0 – this event occurs with probability 0, so you can ignore it. Think of X as taking values in the interval $(0, 1]$.)

(b) Directly from the definition of the c.d.f., $F_Y(y) = \mathbb{P}(Y \le y)$, of the RV Y, show that

$$F_Y(y) = \begin{cases} 0 & \text{if } \ y < 0 \ , \\ 1 - \mathrm{e}^{-y} & \text{if } \ y > 0 \end{cases}$$

(note that I did not write anything about $F_Y(0)$ – since Y is a continuous RV, $\mathbb{P}(Y = 0) = 0$, so that the value of $F_Y(0)$ is not important). You may that the following events are the same:

$$\{Y \le y\} = \{-\ln X \le y\} = \{\ln X \ge -y\} = \{X \ge \mathrm{e}^{-y}\} = \{X < \mathrm{e}^{-y}\}^c \ .$$

(c) Use your result from part (b) to find $f_Y(y)$.

(d) Now use the formula for the p.d.f. of a function, $Y = g(X)$, of a RV, namely

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| \ ,$$

to find $f_Y(y)$ (of course, you should obtain the same result as in part (c)).

Problem 4. The joint p.m.f. $p_{X,Y}(x_k, y_m) = \mathbb{P}(X = x_k, Y = y_m)$ of the discrete RVs X and Y has values given in the table below.

	$Y = 1$	$Y = 3$	$Y = 4$
$X = 5$	0	$\frac{1}{15}$	$\frac{2}{15}$
$X = 7$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(a) Find the marginal p.m.f.s $p_X(x_k)$ and $p_Y(y_m)$.

(b) Find the expected value $\mathbb{E}[X]$ of X.

(c) Compute the conditional p.m.f.s $p_{X|Y}(x_k|y_m) = \mathbb{P}(X = x_k|Y = y_m)$ for $y_m = 1, 3, 4$.

(d) Find the conditional expectations $\mathbb{E}[X|Y = y_m]$ for $y_m = 1, 3, 4$.

(e) The quantity $\mathbb{E}[X|Y]$ can be considered as a random variable which is a function of Y, and written as a linear combination of the indicator functions of the sets $\{Y = y_m\} = Y^{-1}(y_m)$:

$$\mathbb{E}[X|Y] = \sum_m \mathbb{E}[X|Y = y_m] \, I_{Y^{-1}(y_m)} \ .$$

Use your result from part (d) to write $\mathbb{E}[X|Y]$ in this form (using concrete numbers).

(f) Use the representation of $\mathbb{E}[X|Y]$ from part (e) to find its expectation, $\mathbb{E}\left[\mathbb{E}[X|Y]\right]$. According to the so-called *tower rule*, the following equality should hold

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X]$$

(but here you have to compute $\mathbb{E}\left[\mathbb{E}[X|Y]\right]$ directly).

Problem 5. A frog lays Y eggs, where Y is a Poisson RV with parameter $\lambda > 0$:

$$Y \sim \mathrm{Poi}(\lambda) \ ,$$

i.e., the p.m.f. of Y is

$$p_Y(n) = \mathrm{e}^{-\lambda}\,\frac{\lambda^n}{n!} \ , \qquad n = 0, 1, 2, \dots \ ,$$

where $0! := 1$.

Each egg survives independently of the survival of the other eggs, with probability $p \in (0,1)$. From this one can derive that the number X of surviving eggs is a binomial RV with parameters Y and p:

$$X \sim \mathrm{Bin}(Y, p) \ .$$

This means that the RV X, conditioned on the event $\{Y = n\}$, is binomial with parameters n and p:

$$X \,|\, \{Y = n\} \sim \mathrm{Bin}(n, p) \ ,$$

i.e., that the conditional p.m.f. of X conditioned on the value of Y is

$$p_{X|Y}(k|n) = \binom{n}{k}\, p^k\, (1-p)^{n-k} \ , \qquad k = 0, 1, \dots, n \ ,$$

where $\binom{n}{k} := \dfrac{n!}{k!\,(n-k)!}$ are the binomial coefficients.

Show by a direct calculation of the p.m.f. of X that

$$X \sim \mathrm{Poi}(\lambda p) \ .$$

Hint: Use equation (7) derived in Food For Thought Problem 2. You may also find some of the following facts useful:

- series expansion of e^t:
$$\mathrm{e}^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \ , \qquad t \in \mathbb{R} \ ;$$
- binomial formula:
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}\, x^k\, y^{n-k} \ ;$$
- geometric series:
$$\frac{1}{1-q} = \sum_{k=0}^{\infty} q^k \ , \qquad |q| < 1 \ .$$

Food for Thought Problem 1. Let Y be a discrete RV taking values y_m, where the index m runs over a discrete set (finite or infinite). For a given m, consider the pre-image of y_m, i.e., the set

$$Y^{-1}(y_m) = \{\omega \in \Omega \,|\, Y(\omega) = y_m\} \ , \quad \text{which is often written as } \{Y = y_m\} \ . \tag{1}$$

(a) Convince yourself that when k runs over all possible values, the sets $\{Y = y_m\}$ form a partition of the sample space Ω, i.e.,

$$\bigcup_m Y^{-1}(y_m) = \Omega \ , \qquad Y^{-1}(y_m) \cap Y^{-1}(y_n) = \varnothing \ \text{ for } m \neq n \ . \tag{2}$$

(b) Let $A \subset \Omega$ be an event. Recall that the *indicator function*, $I_A : \Omega \to \mathbb{R}$, of an event A is defined as

$$I_A(\omega) = \begin{cases} 0 & \text{if } \ \omega \notin A \ , \\ 1 & \text{if } \ \omega \in A \ . \end{cases}$$

If $I_A^{-1} : \{0,1\} \to \Omega$ stands for the inverse of I_A (of course, I_A^{-1} can take only the numbers 0 or 1 as an argument), then obviously

$$\begin{aligned} I_A^{-1}(0) &= \{\omega \in \Omega \ : \ I_A(\omega) = 0\} = A^c \ , \\ I_A^{-1}(1) &= \{\omega \in \Omega \ : \ I_A(\omega) = 1\} = A \ , \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(I_A^{-1}(0)) &= \mathbb{P}(\{\omega \in \Omega \ : \ I_A(\omega) = 0\}) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A) \ , \\ \mathbb{P}(I_A^{-1}(1)) &= \mathbb{P}(\{\omega \in \Omega \ : \ I_A(\omega) = 1\}) = \mathbb{P}(A) \ . \end{aligned} \tag{3}$$

For a given m, consider the indicator function $I_{Y^{-1}(y_m)} : \Omega \to \{0,1\}$ of the set $Y^{-1}(y_m)$. This is a discrete RV variable satisfying

$$I_{Y^{-1}(y_m)}(\omega) = \begin{cases} 0 & \text{if } \ \omega \notin Y^{-1}(y_m) \, , \ \text{i.e., } \ \omega \notin Y^{-1}(y_m) \ , \\ 1 & \text{if } \ \omega \in Y^{-1}(y_m) \, , \ \text{i.e., } \ \omega \in Y^{-1}(y_m) \ . \end{cases}$$

Convince yourself that the RV Y can be written as a linear combination of the indicator functions of the sets $Y^{-1}(y_m)$ as m runs over all allowed values, as follows:

$$Y = \sum_m y_m \, I_{Y^{-1}(y_m)} \tag{4}$$

(simply note that $I_{Y^{-1}(y_m)}(\omega)$ is equal to 1 exactly if $Y(\omega) = y_m$).

(c) Convince yourself that the p.m.f.

$$p_{I_A}(y) = \mathbb{P}(I_A = y) = \mathbb{P}(I_A^{-1}(y)) \qquad \text{for } \ y \in \{0,1\}$$

of the indicator function I_A is equal to (recall (3))

$$p_{I_A}(y) = \mathbb{P}(I_A^{-1}(y)) = \begin{cases} \mathbb{P}(A^c) = 1 - \mathbb{P}(A) & \text{if } \ y = 0 \ , \\ \mathbb{P}(A) & \text{if } \ y = 1 \ . \end{cases} \tag{5}$$

(d) Directly from the definition of expectation and from (5), show that

$$\mathbb{E}[I_A] = \mathbb{P}(A) \ . \tag{6}$$

(e) Take expectation of both sides of (4) and use the linearity property of expectation (i.e., the fact that $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$) and (6) to obtain

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_m y_m \, I_{Y^{-1}(y_m)}\right] = \sum_m y_m \, \mathbb{E}\left[I_{Y^{-1}(y_m)}\right] = \sum_m y_m \, \mathbb{P}(Y^{-1}(y_m)) = \sum_m y_m \, p_Y(y_m) \ ,$$

which coincides with the definition of $\mathbb{E}[Y]$, as it should.

Food for Thought Problem 2. Let X and Y be discrete RVs taking values x_k and y_m, where the indices k and m run over discrete sets (finite or infinite).

Use the notation (1) and the fact (2) that the sets $\{Y = y_m\}$ form a partition of the sample space Ω, to convince yourselves that the following derivation of the p.m.f. of the RV X through conditioning on the RV Y is correct:

$$\begin{aligned} p_X(x_k) &= \mathbb{P}(X = x_k) = \mathbb{P}(\{X = x_k\} \cap \Omega) \\ &= \mathbb{P}\Big(\{X = x_k\} \cap \bigcup_m \{Y = y_m\}\Big) \\ &= \mathbb{P}\Big(\bigcup_m \{X = x_k, Y = y_m\}\Big) \\ &= \sum_m \mathbb{P}\left(X = x_k, Y = y_m\right) \\ &= \sum_m \mathbb{P}\left(X = x_k \,|\, Y = y_m\right) \, \mathbb{P}(Y = y_m) \\ &= \sum_m \, p_{X|Y}(x_k|y_m) \; p_Y(y_m) \ . \end{aligned} \tag{7}$$