# MATH 5483 Homework 2 Due Thu, Jan 31

**Problem 1.** In some of the parts of this problem, the objects you are asked to construct do not exist – in these cases, explain why (just state an appropriate property, without proofs).

(a) Give an example of a bounded sequence in $\mathbb{R}$ that does not converge.

(b) Give an example of a bounded sequence in $\mathbb{Q}$ that does not converge (to a limit in $\mathbb{Q}$).

(c) Give an example of a bounded and strictly increasing sequence in $\mathbb{R}$ that does not converge.

(d) Give an example of an unbounded sequence in $\mathbb{R}$ that converges.

(e) Give an example of a divergent series $\sum_n a_n$ with $a_n \to 0$.

(f) Give an example of a convergent series $\sum_n a_n$ that is not absolutely convergent (i.e., $\sum_n |a_n|$ does not converge).

(g) Give an example of a divergent alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ with $a_n > 0$ for all $n$, and $a_n \to 0$ as $n \to \infty$.

(h) Give an example of a function $f$ such that $\int_1^{\infty} f(x)\,\mathrm{d}x$ is finite, but $\sum_{n=1}^{\infty} f(n)$ diverges.

(i) Give an example of a function $f$ such that $\sum_{n=1}^{\infty} f(n)$ converges, but $\int_1^{\infty} f(x)\,\mathrm{d}x$ is infinite.

(j) Give an example of a strictly increasing function with a strictly decreasing derivative.

(k) Give an example of a strictly decreasing function with a strictly increasing derivative.

(l) Give an example of an $L^{\infty}(\mathbb{R})$ function that is not in $L^2(\mathbb{R})$.

(m) Give an example of an $L^1(\mathbb{R})$ function that is not in $L^2(\mathbb{R})$.

(n) Give an example of an $L^2(\mathbb{R})$ function that is not in $L^1(\mathbb{R})$.

**Problem 2.** Consider the inner product vector space $L^2(\mathbb{R})$ of functions $f : \mathbb{R} \to \mathbb{C}$ whose $L^2$ norm is finite: $\|f\|_2 < \infty$. Recall that the inner product and the norm are defined as

$$\langle f, g\rangle := \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,\mathrm{d}x\ , \qquad \|f\|_2 := \sqrt{\langle f, f\rangle}\ .$$

(a) Prove the *parallelogram law*,

$$\|f+g\|_2^2 + \|f-g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2\ , \qquad f, g \in L^2\ .$$

Draw a picture to explain the origin of the word "parallelogram".

(b) Prove the so-called "polarization identity",

$$\langle f, g\rangle = \frac{1}{4}\Big(\|f+g\|_2^2 - \|f-g\|_2^2 + i\|f+ig\|_2^2 - i\|f-ig\|_2^2\Big) \ , \qquad f, g \in L^2 \ .$$

**Problem 3.** Recall that the *trace* of a square matrix $A = (a_{ij})$ is defined as the sum of all its diagonal elements:

$$\operatorname{tr} A = \sum_i a_{ii} \ .$$

(a) Consider the $\mathbb{C}$-linear space $\mathbb{C}^{2\times 2}$ of $2 \times 2$ matrices with complex entries. Show that it becomes an inner product linear space with respect to the inner product

$$(A, B) := \operatorname{tr}(A\,B^*) \ ,$$

where $B^*$ is the *adjoint* matrix of $B = (b_{ij})$, i.e., the $(ij)$th entry of $B^*$ is equal to $\overline{b_{ji}}$. In other words, show that $(A, B)$ satisfies all axioms of inner product.

(b) Construct an orthonormal basis in $\mathbb{C}^{2\times 2}$ endowed with the linear product defined in (a).

(c) Modify the inner product defined in part (a) to introduce an inner product in the $\mathbb{C}$-linear space $\mathbb{C}^{2\times 3}$ of all $2 \times 3$ matrices with complex entries. Do not forget to show that $(A, B) = \overline{(B, A)}$.

**Problem 4.** Let $\mathcal{A}$ be the collection of all real-valued functions $f$ that have a Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\, f^{(n)}(0)\, x^n \ ,$$

whose interval of convergence is some interval about the origin (the interval of convergence is allowed to vary with $f$).

Let $D = \frac{\mathrm{d}}{\mathrm{d}x}$ stand for the differentiation operator, and $f(D)$ stand for the differential operator

$$f(D) = \sum_{n=0}^{\infty} \frac{1}{n!}\, f^{(n)}(0)\, D^n = \sum_{n=0}^{\infty} \frac{1}{n!}\, f^{(n)}(0)\, \frac{\mathrm{d}^n}{\mathrm{d}x^n} \ .$$

(a) Show that $\mathcal{A}$ becomes an inner product linear space with respect to the inner product

$$(f, g) = f(D)\, g(x)\big|_{x=0} \ .$$

(b) Show that the family

$$\left\{ \frac{1}{\sqrt{n!}}\, x^n \ : n \in \mathbb{N} \right\}$$

is orthonormal with respect to this inner product.