

Problems 11, 14, 16 from Section 1.3 of the book.

Additional problem 1. The notions of *limit superior* and *limit inferior* of sets are defined and discussed in Section 0.1 of the book. As you recall, for a sequence $\{a_j\}_{j=1}^{\infty}$ of real numbers, *limit superior* and *limit inferior* they are defined as

$$\limsup a_n := \inf_{n \in \mathbb{N}} \sup_{j \geq n} a_j, \quad \liminf a_n := \sup_{n \in \mathbb{N}} \inf_{j \geq n} a_j.$$

In the questions below, (X, \mathcal{M}, μ) is a measure space and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$.

- (a) Let $\{a_j\}_{j=1}^{\infty}$ be a non-decreasing sequence. Prove that $\lim_{j \rightarrow \infty} a_j = \sup_{j \in \mathbb{N}} a_j$ using the ϵ - δ definitions (here the limit can have a finite value or be equal to infinity).
- (b) Show that if $\{a_j\}_{j=1}^{\infty}$ converges, then $\liminf a_j = \limsup a_j = \lim_{j \rightarrow \infty} a_j$.
- (c) Show that $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.
- (d) Give an example in which $\mu(\liminf E_j)$ and $\liminf \mu(E_j)$ are not equal.
- (e) Prove that $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$.
- (f) Give an example in which $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$ and $\mu(\limsup E_j) > \limsup \mu(E_j)$.
- (g) Give an example in which $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \infty$ and $\mu(\limsup E_j) < \limsup \mu(E_j)$.