MATH 5453 Homework 2 Due Thu, Sep 11

Problems 11, 14, 16 from Section 1.3 of the book.

Additional problem 1. The notions of *limit superior* and *limit inferior* of sets are defined and discussed in Section 0.1 of the book. As you recall, for a sequence $\{a_j\}_{j=1}^{\infty}$ of real numbers, *limit superior* and *limit inferior* they are defined as

$$\limsup a_n := \inf_{n\in\mathbb{N}} \sup_{j\geq n} a_j \ , \qquad \liminf a_n := \sup_{n\in\mathbb{N}} \inf_{j\geq n} a_j \ .$$

In the questions below, $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$.

(a) Let $\{a_j\}_{j=1}^{\infty}$ be a non-decreasing sequence. Prove that $\lim\limits_{j\to\infty} a_j = \sup\limits_{j\in\mathbb{N}} a_j$ using the ϵ-δ definitions (here the limit can have a finite value or be equal to infinity).

(b) Show that if $\{a_j\}_{j=1}^{\infty}$ converges, then $\liminf a_j = \limsup a_j = \lim\limits_{j\to\infty} a_j$.

(c) Show that $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.

(d) Give an example in which $\mu(\liminf E_j)$ and $\liminf \mu(E_j)$ are not equal.

(e) Prove that $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$.

(f) Give an example in which $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$ and $\mu(\limsup E_j) > \limsup \mu(E_j)$.

(g) Give an example in which $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \infty$ and $\mu(\limsup E_j) < \limsup \mu(E_j)$.