MATH 4093/5093 Homework 2 Due Fri, 09/10/10

Problem 1.

(a) Show that the function $y(t)$ defined implicitly by the equation

$$y(t)^2 - 2t\mathrm{e}^{-y(t)} - 4t^2 - 1 = 0 \tag{1}$$

satisfies the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{4t + \mathrm{e}^{-y}}{y + t\mathrm{e}^{-y}} \ , \qquad t \geq 0 \ , \\ y(0) &= 1 \ . \end{aligned} \tag{2}$$

Hint: To derive the differential equation for $y(t)$, differentiate (1) with respect to t.

(b) I used a method for solving nonlinear equations to find that

$$y(\tfrac{1}{2}) = 1.4916441946682422359111377449738198852929380037924846629297190060 5 \ .$$

To find this value, I plugged $t = \frac{1}{2}$ in (1) to get $y(\frac{1}{2})^2 - \mathrm{e}^{-y(\frac{1}{2})} - 2 = 0$, and solved this equation numerically (on Mathematica) using an extremely fast method for solving nonlinear equations called the *Newton method*. I hope that we will have a chance to talk about it later this semester.

Go to the web-site

`http://www.pcs.cnu.edu/~bbradie/matlab.html` ,

click on "Initial Value Problems", and from the web-site this will take you download the Matlab code `euler.m`, which solves an IVP for a system of ODEs by Euler's method (this is the same file that is in your notes). Write a Matlab file containing the right-hand side of the differential equation from the IVP (2).

Solve the IVP (2) and find $y(\frac{1}{2})$ numerically, using Euler's method. Do it with $N =$ 10, 100, 1000, 10000, and 100000 (which corresponds to stepsize $h =$ 0.05, 0.005, 0.0005, 0.00005 and 0.000005, respectively). In a table, put the values of N, the corresponding values of $y(\frac{1}{2})_{\text{approx}}$ obtained by running `euler.m`, as well as the absolute errors $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, where $y(\frac{1}{2})_{\text{exact}}$ is the exact value. Use small enough tolerance, i.e., 10^{-12}. (There is no need to do the table in Matlab – if you want, do it in Matlab, if not, just write it by hand.)

Note that the code `euler.m` can solve an IVP for a system of ODEs; it finds the number of equations in your system from the number of initial conditions you give it. In this problem, the initial condition you will give to the Matlab program is only one, so the code will understand that you are solving an IVP for a single ODE.

(c) Plot by hand or using some software the logarithm of the error, $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the approximate straight line that goes through these points. How does the value of this slope match with the theoretical prediction for the value of the error of Euler's method?

Note that you can use natural logarithms or logarithms base 10, or any other base to plot the results (but use the same base for both axes!) – this is not going to change the slope of the approximate straight line.

Plotting in Matlab is very easy – for example, to plot the points (x_j, y_j) (for $j = 1, 2, 3, 4, 5$), where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (4, 5, 6, 7, 8)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5) = (12, 11, 12, 10, 9)$, you can type the following Matlab commands (here `>` stands for the Matlab prompt):

```
> x = [4 5 6 7 8]
> y = [12 11 12 10 9]
> plot(x,y)
```

If, instead of `plot(x,y)` you type `plot(x,y,'-*')`, the data will be represented by stars connected by straight lines.

Problem 2. Consider the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= y^2 + \frac{1}{t^2} , \qquad t \in [1, 2] , \\ y(1) &= -\frac{1}{2} . \end{aligned} \tag{3}$$

(a) Develop Taylor's method of order 2 to solve the IVP (3).

Hint: In other words, find the derivative of the right-hand side of the ODE in (3) (which represents the second derivative $y''(t)$ of the exact solution), and write expressions for the values of w_0 and w_{i+1} (where w_{i+1} should be expressed in terms of the values of w_i, t_i and the stepsize h.

Hint: I obtained that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t, y(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)^2 + \frac{1}{t^2}\right) = 2y^3(t) + \frac{2y(t)}{t^2} - \frac{2}{t^3} ,$$

and, using this, I obtained the following expression for w_{i+1}:

$$w_{i+1} = w_i + h\left(w_i^2 + \frac{1}{t_i^2}\right) + \frac{h^2}{2}\left(2w_i^3 + \frac{2w_i}{t_i^2} - \frac{2}{t_i^3}\right) ;$$

I want to see your detailed derivations of these results.

(b) Use Bradie's code `taylor2nd.m` (which can be downloaded from the same web-site as `euler.m`) and a Matlab file that will supply the right-hand side of the ODE and the first derivative of the right-hand side of the ODE with respect to t, to solve the IVP (3) by using Taylor method of order 2 in order to find the value of $y(2)$, with $N = 10$, 100, 1000, and 10000. Record the numerical values of $y(2)$ obtained by using different N.

Hint: The Matlab file `fun1_and_fun1der.m` that I used to compute the value of the right-hand side and its derivative for one of the examples considered in class was

```
function [fun1, fun1der] = fun1_and_fun1der(t,x)

     fun1 = 1 + x/t;
     fun1der = 1/t;
```

(c) Plot by hand or using some software the logarithm of the error, $|y(2)_{\text{exact}} - y(2)_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP (3) is

$$y(t)_{\text{exact}} = \tfrac{1}{2t}\left[\sqrt{3}\tan\left(\tfrac{\sqrt{3}}{2}\ln|t|\right) - 1\right] . \tag{4}$$

Problem 3. Consider the same IVP as in Problem 3.

(a) Develop Taylor's method of order 3 to solve the IVP (3).

Hint: Here is what I obtained for the second derivative of $f(t, y(t))$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t, y(t)) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(y(t)^2 + \frac{1}{t^2}\right) = 6y^4(t) + \frac{8y^2(t)}{t^2} - \frac{4y(t)}{t^3} + \frac{8}{t^4} .$$

(b) Look at Bradie's codes `taylor2nd.m` and `taylor4th.m` (which can be downloaded from the same web-site as `euler.m`) and write your code `taylor2rd.m` that solves IVPs for ODEs using Taylor's method of order 3. Perhaps the easiest way to do this will be to take `taylor4th.m` and to modify it (the modification will be really minor). Do not forget to rename the file to `taylor3rd.m` as well as to the change first line of the file to `function [wi, ti] = taylor3rd ( RHS, t0, x0, tf, N )` – otherwise Matlab will complain.

Write a Matlab file that will the right-hand side of the ODE and all the derivatives that are needed by `taylor3rd.m`.

Please attach a printout of the two Matlab files!

(c) Run `taylor3rd.m` to solve the IVP (3) with $N = 10$, 100, 1000, and 10000. Record the numerical values of $y(2)$ obtained by using different N.

(d) Plot by hand or using some software the logarithm of the error, $|y(2)_{\text{exact}} - y(2)_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP is given in (4).