MATH 4513 Homework 2 Due Fri, Sep 11, 2015

Problem 1. [Thinking simply, again]
Find the exact value of the number γ defined by the following expression:

$$\gamma = \cfrac{1}{1 + \cfrac{\mathbf{1}}{\mathbf{1} + \cfrac{\mathbf{1}}{\mathbf{1} + \cfrac{\mathbf{1}}{\mathbf{1} + \cdots}}}} \ . \tag{1}$$

Hint: Look at the part of the right-hand side of (1) written with bold face digits. How it is related to γ?

Problem 2. [Existence and uniqueness of roots of algebraic equations]

(a) State the Intermediate Value Theorem (you can find it, for example, on page 89 of the 7th edition of Stewart's *Calculus*).

(b) Consider the equation

$$f(x) = x - \cos x = 0 \ . \tag{2}$$

Prove that (2) has a solution in the interval $[0, \frac{\pi}{2}]$. Explain your reasoning in detail.

(c) Prove that the solution of (2) in the interval $[0, \frac{\pi}{2}]$ is unique.

Hint: What can you say about the derivative of the function f on the interval $[0, \frac{\pi}{2}]$?

Problem 3. [Bifurcation in a logistic equation with constant harvesting]
Recall the logistic equation with constant harvesting,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \alpha P \left(1 - \frac{P}{K}\right) - H$$

where P is the population, t is the time, α is the reproduction rate of the species, K is the carrying capacity of the ecosystem, and H is the harvesting rate. After the non-dimensionalization $y := P/K$, $\tau := \alpha t$, $h := H/(\alpha K)$, the equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = y(1-y) - h \ . \tag{3}$$

Think of y and h as positive quantities.

(a) Rewrite the condition $y(1-y) - h = 0$ for a FP of (3) in the form $f(y) = g(y)$ with $f(y) = y(1-y)$ and $g(y) = h$. Plot the graph of $f(y)$ for $y > 0$. What is the maximum value of f? For which value of y is it attained?

(b) Draw on the same plot the graphs of $f(y)$ and $g(y)$. How many fixed points of (3) are there for different values of h? For which value h_0 of the parameter h does bifurcation occur? What is the value of the fixed point y_0^* at this value of h_0?

(c) What kind of bifurcation occurs in the system (3)? Justify your answer by writing down the Taylor expansion of the right-hand side of (3) considered as a function of the variables y and h, in a Taylor series around $(y_0^*; h_0)$. What can you say about the stability of the fixed points?

(d) Draw the bifurcation diagram of (3), i.e., the position of the fixed point(s) as a function of the parameter h. Write down the (implicit) equation of the curve on the bifurcation diagram. Don't forget to denote the positions of the stable FPs with a solid line, and the position of the unstable ones with a dashed line.

Problem 4. [Bifurcation in a logistic equation with linear harvesting]

Assume that the harvesting is a linear function of the population, i.e., consider the following modification of the logistic equation:

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = y(1-y) - (a+by) \ , \tag{4}$$

where a and b are positive parameters.

(a) Rewrite the condition $y(1-y) - (a+by) = 0$ for a FP of (4) in the form $f(y) = g(y)$ with $f(y) = y(1-y)$ and $g(y) = a + by$. Plot the graphs of $f(y)$ and $g(y)$ together, for three cases: when (4) has no FP, when (4) has exactly one FP, and when (4) has two FPs.

(b) Write down the conditions for the equation (4) to have exactly one FP. Solve them to obtain a relation between the parameters a and b.

Hint: Recall that the graphs of $f(y)$ and $g(y)$ must "touch" at a point, which gives you two conditions.

(c) Plot the relation obtained in part (b) in the (a, b) plane, and indicate how many FPs of (4) are there in each region in your plot.