MATH 4193/5103 **Homework 2** **Due Tue, Feb 26, 2013**

Problem 1. [A model of a fishery]

The equation

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - H\frac{N}{A+N} \tag{1}$$

provides a simple model of a fishery. Here $N(t) \geq 0$ is the population of fish at time t, $R = \text{const} > 0$ is the reproduction rate, $K = \text{const} > 0$ is the carrying capacity of the system, $H = \text{const} > 0$ characterizes the intensity of fishing, and $A = \text{const} > 0$ is another positive constant. In the absence of fishing, the population of fish evolves logistically, which is reflected by the term $RN\left(1 - \frac{N}{K}\right)$ in the right-hand side of (1). The term $-H\frac{N}{A+N}$ in the right-hand side of (1) models the effects of fishing. The choice of this particular form of the "fishing" term made because (i) it is simple, (ii) the model has a fixed point at $N = 0$ for all values of the parameters, as it should be, and (iii) it is reasonable to assume that the rage at which fish are caught increases with N, and for large N it "saturates" at H.

(a) Show that the system (1) can be written in dimensionless form as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-x) - h\frac{x}{a+x}$$

for suitably defined dimensionless quantities x, τ, a, and h. (Write down explicitly the relations between the original quantities and the dimensionless ones.)

(b) Show that the system can have one, two, or three fixed points, depending on the values of a and h. Classify the stability of the fixed points in each case.

(c) Analyze the dynamics of the system near $x = 0$ and show that a bifurcation occurs when $h = a$. What kind of bifurcation is it?

(d) Show that another bifurcation occurs when $h = \frac{1}{4}(a+1)^2$, for $a < a_c$, where a_c is some "critical" value. What is the value of a_c? Classify this bifurcation.

(e) Plot the stability diagram of the system in the (a, h) parameter space. Can hysteresis occur in any of the stability regions?

Problem 2. [Dynamics on a circle]

Consider the interval $[-\pi, \pi]$ (with its ends identified) as a model of the circle S. Define the two-parameter family of functions $f : S \to \mathbb{R}$ by

$$f(\theta) = \omega - a + \frac{a}{\pi}|\theta| \ . \tag{2}$$

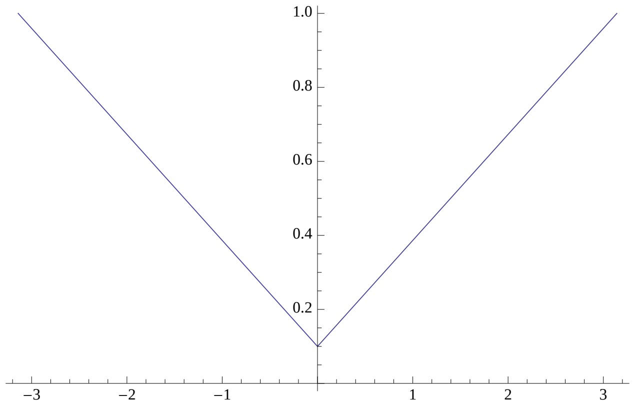

Figure 1: The graph of the function (2) for $\omega = 1$ and $a = 0.9$.

This function is piece-wise linear (i.e., its graph consists of segments of straight lines), and satisfies $f(-\pi) = f(\pi) = \omega$, $f(0) = \omega - a$; see Figure 1. For simplicity, in all parts of this problem assume that $\boxed{\omega \geq 0}$.

Consider the system

$$\dot{\theta} = \omega + a|\theta| \ , \tag{3}$$

where $\theta : \mathbb{R} \to S$ is an unknown function.

(a) For each value of $\omega \geq 0$, find an explicit expression for the value of a for which the system (3) undergoes a bifurcation. What kind of bifurcation is it?

(b) For a given value of $\omega \geq 0$, and for a value of a in the range in which the system (2) has exactly two fixed points, θ_1^* and θ_2^* (assume that $\theta_1^* < \theta_2^*$) find the values of θ_1^* and θ_2^* expressed in terms of the values of ω and a. Plot these values in the (a, θ^*) plane for a given value of $\omega \geq 0$. Indicate the value of ω in the (a, θ^*) plane. Use a solid line to denote the stable fixed point and a dashed line to denote the unstable fixed point in the (a, θ^*) plane.

(c) In the (ω, a) plane, indicate the region in which the system (2) has two fixed points, and the region where it has no fixed points.

(d) For given values of $\omega \geq 0$ and a such that the system (2) has no fixed points, find the period T as a function of ω and a. For a given $\omega \geq 0$, sketch the graph of T vs. a.

Problem 3. [Solution of a constant-coefficient linear system as an exponential]

If M is a square $m \times m$ matrix (i.e., a matrix of size $m \times m$ with real or complex entries, one can define the *exponential* of M as

$$e^{\mathsf{M}} \equiv \exp \mathsf{M} := \sum_{j=0}^{\infty} \frac{1}{j!} \mathsf{M}^j \ , \tag{4}$$

where M^0 is by definition the identity matrix I. It can be shown that this series converges for any square matrix M.

Exponentials of matrices are useful for representing the solutions of initial-value problems for systems of linear ordinary differential coefficients with constant coefficients,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \mathsf{A}\mathrm{x}\ , \qquad t \in [0, \infty) \\ \mathbf{x}(0) &= \mathbf{b}\ . \end{aligned} \tag{5}$$

(a) Directly from the definition (4), show that $\mathsf{M}\mathrm{e}^{\mathsf{M}} = \mathrm{e}^{\mathsf{M}}\mathsf{M}$ for any square matrix M.

(b) Let A be a given $m \times m$ matrix, and t be a real number. Then one can think of $\mathrm{e}^{\mathsf{A}t}$ as a function taking an argument from $\mathbb{R}$ and having values in the $m \times m$ matrices. Directly from (4), show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathsf{A}t} = \mathsf{A}\mathrm{e}^{\mathsf{A}t}$ and $\mathrm{e}^{\mathsf{A}t}|_{t=0} = \mathsf{I}$.

(c) Use your result from part (b) to show that the solution of the initial-value problem (5) can be written as

$$\mathbf{x}(t) = \mathrm{e}^{\mathsf{A}t}\mathbf{b}\ .$$

(d) For any positive real numbers s and t show that $\mathrm{e}^{\mathsf{A}s}\mathrm{e}^{\mathsf{A}t} = \mathrm{e}^{\mathsf{A}(s+t)}$ and use this to show that $\mathbf{x}(t+s) = \mathrm{e}^{\mathsf{A}s}\mathbf{x}(t)$. How can you interpret this result "physically"?

(e) Directly from the definition (4), show that

$$\mathrm{e}^{\mathsf{T}\mathsf{B}\mathsf{T}^{-1}} = \mathsf{T}\mathrm{e}^{\mathsf{B}}\mathsf{T}^{-1}\ .$$

(f) Compute $\mathrm{e}^{\mathsf{B}t}$ for $\mathsf{B} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

(g) Rewrite the linear system

$$\begin{aligned} \dot{x} &= 2x \\ \dot{y} &= 3x - y \end{aligned} \tag{6}$$

in a matrix form as $\dot{\mathbf{x}} = \mathsf{A}\mathbf{x}$. If $\mathsf{T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with inverse $\mathsf{T}^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, find $\mathsf{B} = \mathsf{T}^{-1}\mathsf{A}\mathsf{T}$.

(h) Use your results from the previous part of this problem to write down $\mathrm{e}^{\mathsf{A}t}$ (where A is the matrix from the right-hand side of (6)).

(i) Use your result from part (h) to write down the solution of the initial-value problem consisting of the system (6) and the initial condition $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.