MATH 4193/5103 **Homework 2** **Due 2/18/20 (Tuesday)**

Problem 1. [Fixed points and their stability without calculations]

Consider the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) := x(x-2)^2(x-4)^5 \ . \tag{1}$$

(a) Without doing any calculations, sketch the right-hand side of the ODE (1) in the phase plane (i.e., plot the graph of the function f from (1) in the (x, x') plane). Find all the fixed points of the ODE (1). Indicate with arrows on the x-axis to the left or to the right in which direction will the function $x(t)$ evolve for different initial conditions. No explanation is needed.

(b) Based on your observations in part (a), classify the fixed points as stable (attracting), unstable (repelling) or semi-stable, and put them on the x-axis (full circle, empty circle, or half-full circle, respectively).

(c) Without taking any derivatives I was able to figure out that:

- the Taylor series expansion of the function $f(x)$ from (1) about the point 0 has the form
$$f(x) = -4096\, x + [\text{higher order terms in } x] \ ;$$
- the Taylor series expansion of the function $f(x)$ from (1) about the point 2 has the form
$$f(x) = -64\,(x-2)^2 + [\text{higher order terms in } (x-2)] \ .$$

Without doing any calculations, write down the first term in the Taylor expansion of the function $f(x)$ from (1) about the point 4. Explain clearly how you computed the coefficient.

(d) Sketch in the (t, x)-plane the solutions starting at the following initial conditions: $x(0) = -1, 0, 1, 2, 3, 4, 5$ (sketch all solutions on the same graph). In each of these cases, find the asymptotic behavior of the solution $x(t)$, i.e., determine $\lim_{t\to\infty} x(t)$.

Problem 2. [A simple bifurcation diagram]

Consider the one-parameter family of ODEs

$$x' = f_\mu(x) := x(x-\mu) \ . \tag{2}$$

(a) Determine all the fixed points of the ODE (2).

(b) Determine the stability of the fixed points of the ODE (2) for $\mu < 0$. Sketch the graph of f in the phase plane in this case.

(c) Determine the stability of the fixed points of the ODE (2) for $\mu > 0$. Sketch the graph of f in the phase plane in this case.

(d) Plot the bifurcation diagram of the ODE (2), i.e., the position of the fixed points as functions of μ. Indicate the attracting fixed point with a solid line and the repelling fixed point with a dashed line.

Problem 3. [Saddle-node (tangent, blue sky) bifurcation in a 1-parameter family]

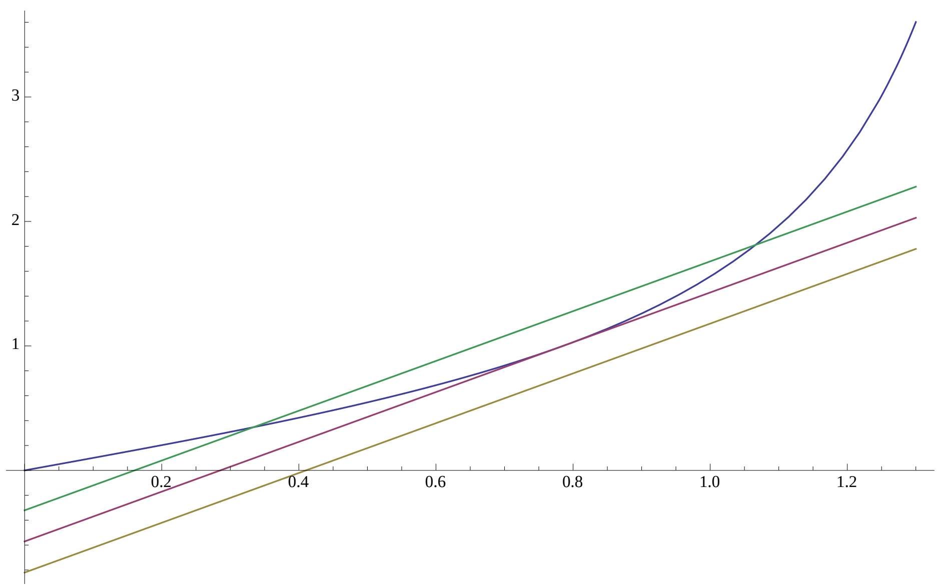
Consider the one-parameter family of ODEs

$$x' = f_\mu(x) := \tan x - 2x - \mu \ , \tag{3}$$

where μ is a parameter, and the solution $x(t)$ can take only positive values. Your goal in this problem is to find the value μ_c of the parameter μ such that for $\mu < \mu_c$ the ODE has no fixed points, while for $\mu > \mu_c$ the ODE has two fixed points of opposite stability. Please follow the steps below.

(a) Rewrite the function $f_\mu(x)$ from (3) as a difference of the functions $\phi(x) = \tan x$ and $\psi_\mu(x) = 2x + \mu$.

(b) The graph below shows the graphs of $\phi(x)$ and $\psi_\mu(x)$ for three different values of μ. At the critical value μ_c of the parameter μ above which the ODE has fixed points, the



graphs of $\phi(x)$ and $\psi_\mu(x)$ are tangent to each other. Write the system of two equations for the unknowns μ_c and the value x_c^* of x for which these two graphs are tangent. Write explicitly what the meaning of each of these two equations is.

(c) Solve the two equations derived in part (b) to find the values of μ_c and x_c^*.

(d) Expand the function $f_\mu(x)$ (considered as a function of two independent variables, μ and x), in a Taylor series about the point (μ_c, x_c^*). Since the dependence of $f_\mu(x)$ on μ is very simple, you will not need to use the formula for Taylor series for a function of two variables – it will be enough to use the Taylor expansion of the function $\tan x$ about an appropriately chosen point. For your convenience, here are the Taylor expansions of $\tan x$ about 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ – use the one that you need:

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

$$\tan x = \tfrac{1}{\sqrt{3}} + \tfrac{4}{3}\left(x - \tfrac{\pi}{6}\right) + \tfrac{4}{3\sqrt{3}}\left(x - \tfrac{\pi}{6}\right)^2 + \tfrac{8}{9}\left(x - \tfrac{\pi}{6}\right)^3 + \tfrac{4}{3\sqrt{3}}\left(x - \tfrac{\pi}{6}\right)^4 + \cdots$$

$$\tan x = 1 + 2\left(x - \tfrac{\pi}{4}\right) + 2\left(x - \tfrac{\pi}{4}\right)^2 + \tfrac{8}{3}\left(x - \tfrac{\pi}{4}\right)^3 + \tfrac{10}{3}\left(x - \tfrac{\pi}{4}\right)^4 + \cdots$$

$$\tan x = \sqrt{3} + 4\left(x - \tfrac{\pi}{3}\right) + 4\sqrt{3}\left(x - \tfrac{\pi}{3}\right)^2 + \tfrac{40}{3}\left(x - \tfrac{\pi}{3}\right)^3 + \tfrac{44}{\sqrt{3}}\left(x - \tfrac{\pi}{3}\right)^4 + \cdots$$

(e) Take the lowest-order terms that contain μ and x in the Taylor expansion you obtained in part (d), and use them to find an approximate expression for the fixed points as functions of the difference $(\mu - \mu_c)$, for $\mu > \mu_c$. You will obtain that

$$x_{1,2}^* \approx \frac{\pi}{4} \pm \sqrt{\tfrac{1}{2}(\mu - \mu_c)}\ ;$$

I want to see your detailed calculations.

(f) For $\mu > \mu_c$, determine which of the fixed points x_1^* and x_2^* is attracting and which one is repelling. You may use a graph or a calculation (using the graph is easier).

(g) Sketch the bifurcation diagram for the 1-parameter family of ODEs (3), near the point (x_c^*, μ_c).

Problem 4. [Bifurcation in a logistic equation with linear harvesting]

Consider a population $X(T)$ that changes according to the logistic equation and in addition is subjected to a linear harvesting, i.e., in each time interval, a part of the population is removed by harvesting, and the harvesting is assumed to be a linear function of the population at that moment. If R is the reproduction rate of the population, K is the carrying capacity, and A and B are parameters that define the harvesting, the ordinary differential equation governing the evolution of the population is

$$\frac{\mathrm{d}X}{\mathrm{d}T} = RX\left(1 - \frac{X}{K}\right) - (A + BX)\ . \tag{4}$$

(a) It looks like the equation (4) has four parameters, but in fact two of them can be eliminated by a change of variables. Change the independent variable T and the dependent variable X to the new "time" t and "population" x by

$$t = RT\ , \qquad x = \frac{X}{K}\ ,$$

and show that the four-parameter family (4) becomes the two-parameter family

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) - (a+bx) \ . \tag{5}$$

Express the new parameters, a and b, in therms of the old parameters R, K, A, and B.

(b) Rewrite the condition $x(1-x)-(a+bx)=0$ for a FP of (5) in the form $f(x)=g_{a,b}(x)$ with $f(x)=x(1-x)$ and $g_{a,b}(x)=a+bx$. Plot the graphs of $f(x)$ and $g_{a,b}(x)$ together, for three cases: when (5) has no FP, when (5) has exactly one FP, and when (5) has two FPs.

(c) Write down the conditions for the equation (5) to have exactly one FP. Solve them to obtain a relation between the parameters a and b.

Hint: Recall that the graphs of $f(x)$ and $g_{a,b}(x)$ must "touch" at a point, which gives you two conditions.

(d) Plot the relation obtained in part (c) in the (a,b) plane, and indicate how many FPs of (5) are there in each region in your plot.