

**Problem 1. [Fixed points and their stability without calculations]**

Consider the ODE

$$\frac{dx}{dt} = f(x) := x(x-2)^2(x-4)^5. \quad (1)$$

- (a) Without doing any calculations, sketch the right-hand side of the ODE (1) in the phase plane (i.e., plot the graph of the function  $f$  from (1) in the  $(x, x')$  plane). Find all the fixed points of the ODE (1). Indicate with arrows on the  $x$ -axis to the left or to the right in which direction will the function  $x(t)$  evolve for different initial conditions. No explanation is needed.
- (b) Based on your observations in part (a), classify the fixed points as stable (attracting), unstable (repelling) or semi-stable, and put them on the  $x$ -axis (full circle, empty circle, or half-full circle, respectively).
- (c) Without taking any derivatives I was able to figure out that:

- the Taylor series expansion of the function  $f(x)$  from (1) about the point 0 has the form

$$f(x) = -4096x + [\text{higher order terms in } x];$$

- the Taylor series expansion of the function  $f(x)$  from (1) about the point 2 has the form

$$f(x) = -64(x-2)^2 + [\text{higher order terms in } (x-2)].$$

Without doing any calculations, write down the first term in the Taylor expansion of the function  $f(x)$  from (1) about the point 4. Explain clearly how you computed the coefficient.

- (d) Sketch in the  $(t, x)$ -plane the solutions starting at the following initial conditions:  $x(0) = -1, 0, 1, 2, 3, 4, 5$  (sketch all solutions on the same graph). In each of these cases, find the asymptotic behavior of the solution  $x(t)$ , i.e., determine  $\lim_{t \rightarrow \infty} x(t)$ .

**Problem 2. [A simple bifurcation diagram]**

Consider the one-parameter family of ODEs

$$x' = f_\mu(x) := x(x - \mu). \quad (2)$$

- (a) Determine all the fixed points of the ODE (2).
- (b) Determine the stability of the fixed points of the ODE (2) for  $\mu < 0$ . Sketch the graph of  $f$  in the phase plane in this case.

- (c) Determine the stability of the fixed points of the ODE (2) for  $\mu > 0$ . Sketch the graph of  $f$  in the phase plane in this case.
- (d) Plot the bifurcation diagram of the ODE (2), i.e., the position of the fixed points as functions of  $\mu$ . Indicate the attracting fixed point with a solid line and the repelling fixed point with a dashed line.

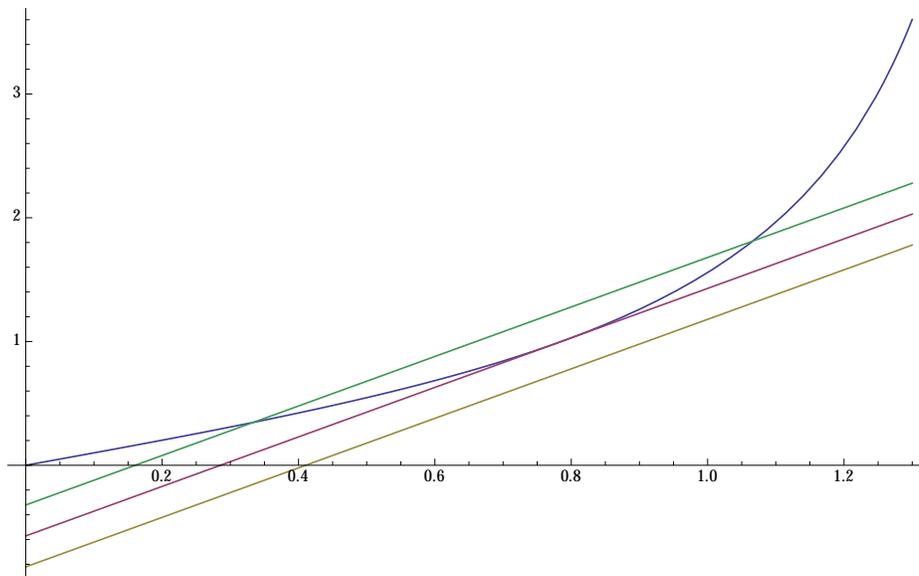
**Problem 3. [Saddle-node (tangent, blue sky) bifurcation in a 1-parameter family]**

Consider the one-parameter family of ODEs

$$x' = f_\mu(x) := \tan x - 2x - \mu, \tag{3}$$

where  $\mu$  is a parameter, and the solution  $x(t)$  can take only positive values. Your goal in this problem is to find the value  $\mu_c$  of the parameter  $\mu$  such that for  $\mu < \mu_c$  the ODE has no fixed points, while for  $\mu > \mu_c$  the ODE has two fixed points of opposite stability. Please follow the steps below.

- (a) Rewrite the function  $f_\mu(x)$  from (3) as a difference of the functions  $\phi(x) = \tan x$  and  $\psi_\mu(x) = 2x + \mu$ .
- (b) The graph below shows the graphs of  $\phi(x)$  and  $\psi_\mu(x)$  for three different values of  $\mu$ . At the critical value  $\mu_c$  of the parameter  $\mu$  above which the ODE has fixed points, the



graphs of  $\phi(x)$  and  $\psi_\mu(x)$  are tangent to each other. Write the system of two equations for the unknowns  $\mu_c$  and the value  $x_c^*$  of  $x$  for which these two graphs are tangent. Write explicitly what the meaning of each of these two equations is.

- (c) Solve the two equations derived in part (b) to find the values of  $\mu_c$  and  $x_c^*$ .

- (d) Expand the function  $f_\mu(x)$  (considered as a function of two independent variables,  $\mu$  and  $x$ ), in a Taylor series about the point  $(\mu_c, x_c^*)$ . Since the dependence of  $f_\mu(x)$  on  $\mu$  is very simple, you will not need to use the formula for Taylor series for a function of two variables – it will be enough to use the Taylor expansion of the function  $\tan x$  about an appropriately chosen point. For your convenience, here are the Taylor expansions of  $\tan x$  about  $0$ ,  $\frac{\pi}{6}$ ,  $\frac{\pi}{4}$ , and  $\frac{\pi}{3}$  – use the one that you need:

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

$$\tan x = \frac{1}{\sqrt{3}} + \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{4}{3\sqrt{3}}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3 + \frac{4}{3\sqrt{3}}\left(x - \frac{\pi}{6}\right)^4 + \dots$$

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + \dots$$

$$\tan x = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + 4\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{3}\right)^3 + \frac{44}{\sqrt{3}}\left(x - \frac{\pi}{3}\right)^4 + \dots$$

- (e) Take the lowest-order terms that contain  $\mu$  and  $x$  in the Taylor expansion you obtained in part (d), and use them to find an approximate expression for the fixed points as functions of the difference  $(\mu - \mu_c)$ , for  $\mu > \mu_c$ . You will obtain that

$$x_{1,2}^* \approx \frac{\pi}{4} \pm \sqrt{\frac{1}{2}(\mu - \mu_c)} ;$$

I want to see your detailed calculations.

- (f) For  $\mu > \mu_c$ , determine which of the fixed points  $x_1^*$  and  $x_2^*$  is attracting and which one is repelling. You may use a graph or a calculation (using the graph is easier).
- (g) Sketch the bifurcation diagram for the 1-parameter family of ODEs (3), near the point  $(x_c^*, \mu_c)$ .

#### Problem 4. [Bifurcation in a logistic equation with linear harvesting]

Consider a population  $X(T)$  that changes according to the logistic equation and in addition is subjected to a linear harvesting, i.e., in each time interval, a part of the population is removed by harvesting, and the harvesting is assumed to be a linear function of the population at that moment. If  $R$  is the reproduction rate of the population,  $K$  is the carrying capacity, and  $A$  and  $B$  are parameters that define the harvesting, the ordinary differential equation governing the evolution of the population is

$$\frac{dX}{dT} = RX \left(1 - \frac{X}{K}\right) - (A + BX) . \quad (4)$$

- (a) It looks like the equation (4) has four parameters, but in fact two of them can be eliminated by a change of variables. Change the independent variable  $T$  and the dependent variable  $X$  to the new “time”  $t$  and “population”  $x$  by

$$t = RT , \quad x = \frac{X}{K} ,$$

and show that the four-parameter family (4) becomes the two-parameter family

$$\frac{dx}{dt} = x(1-x) - (a+bx) . \quad (5)$$

Express the new parameters,  $a$  and  $b$ , in terms of the old parameters  $R$ ,  $K$ ,  $A$ , and  $B$ .

- (b) Rewrite the condition  $x(1-x) - (a+bx) = 0$  for a FP of (5) in the form  $f(x) = g_{a,b}(x)$  with  $f(x) = x(1-x)$  and  $g_{a,b}(x) = a+bx$ . Plot the graphs of  $f(x)$  and  $g_{a,b}(x)$  together, for three cases: when (5) has no FP, when (5) has exactly one FP, and when (5) has two FPs.
- (c) Write down the conditions for the equation (5) to have exactly one FP. Solve them to obtain a relation between the parameters  $a$  and  $b$ .
- Hint:* Recall that the graphs of  $f(x)$  and  $g_{a,b}(x)$  must “touch” at a point, which gives you two conditions.
- (d) Plot the relation obtained in part (c) in the  $(a, b)$  plane, and indicate how many FPs of (5) are there in each region in your plot.