

Problem 1. [Fixed points and their stability without calculations]

Consider the ODE

$$\frac{dx}{dt} = f(x) := x(x-2)^2(x-4)^5. \quad (1)$$

- (a) Without doing any calculations, sketch the right-hand side of the ODE (1) in the phase plane (i.e., plot the graph of the function f from (1) in the (x, x') plane). Find all the fixed points of the ODE (1). Indicate with arrows on the x -axis to the left or to the right in which direction will the function $x(t)$ evolve for different initial conditions. No explanation is needed.
- (b) Based on your observations in part (a), classify the fixed points as stable (attracting), unstable (repelling) or semi-stable, and put them on the x -axis (full circle, empty circle, or half-full circle, respectively).
- (c) Without taking any derivatives I was able to figure out that:

- the Taylor series expansion of the function $f(x)$ from (1) about the point 0 has the form

$$f(x) = -4096x + [\text{higher order terms in } x] ;$$

- the Taylor series expansion of the function $f(x)$ from (1) about the point 2 has the form

$$f(x) = -64(x-2)^2 + [\text{higher order terms in } (x-2)] .$$

Without doing any calculations, write down the first term in the Taylor expansion of the function $f(x)$ from (1) about the point 4. Explain clearly how you computed the coefficient.

- (d) Sketch in the (t, x) -plane the solutions starting at the following initial conditions: $x(0) = -1, 0, 1, 2, 3, 4, 5$ (sketch all solutions on the same graph). In each of these cases, find the asymptotic behavior of the solution $x(t)$, i.e., determine $\lim_{t \rightarrow \infty} x(t)$.

Problem 2. [A simple bifurcation diagram]

Consider the one-parameter family of ODEs

$$x' = f_\mu(x) := x(x - \mu) . \quad (2)$$

- (a) Determine all the fixed points of the ODE (2).
- (b) Determine the stability of the fixed points of the ODE (2) for $\mu < 0$. Sketch the graph of f in the phase plane in this case.

- (c) Determine the stability of the fixed points of the ODE (2) for $\mu > 0$. Sketch the graph of f in the phase plane in this case.
- (d) Plot the bifurcation diagram of the ODE (2), i.e., the position of the fixed points as functions of μ . Indicate the attracting fixed point with a solid line and the repelling fixed point with a dashed line.

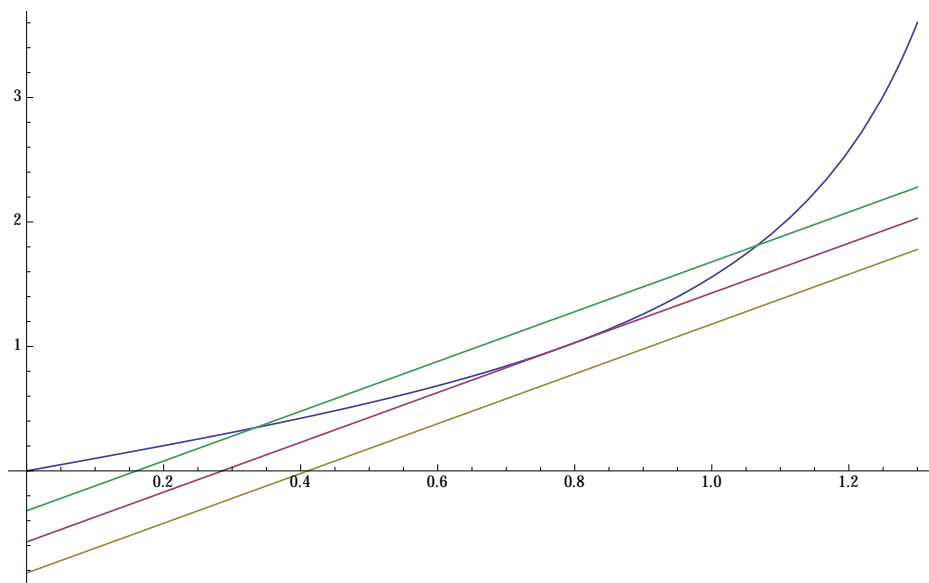
Problem 3. [Saddle-node (tangent, blue sky) bifurcation in a 1-parameter family]

Consider the one-parameter family of ODEs

$$x' = f_\mu(x) := \tan x - 2x - \mu, \quad (3)$$

where μ is a parameter, and the solution $x(t)$ can take only positive values. Your goal in this problem is to find the value μ_c of the parameter μ such that for $\mu < \mu_c$ the ODE has no fixed points, while for $\mu > \mu_c$ the ODE has two fixed points of opposite stability. Please follow the steps below.

- (a) Rewrite the function $f_\mu(x)$ from (3) as a difference of the functions $\phi(x) = \tan x$ and $\psi_\mu(x) = 2x + \mu$.
- (b) The graph below shows the graphs of $\phi(x)$ and $\psi_\mu(x)$ for three different values of μ . At the critical value μ_c of the parameter μ above which the ODE has fixed points, the



graphs of $\phi(x)$ and $\psi_\mu(x)$ are tangent to each other. Write the system of two equations for the unknowns μ_c and the value x_c^* of x for which these two graphs are tangent. Write explicitly what the meaning of each of these two equations is.

- (c) Solve the two equations derived in part (b) to find the values of μ_c and x_c^* .

- (d) Expand the function $f_\mu(x)$ (considered as a function of two independent variables, μ and x), in a Taylor series about the point (μ_c, x_c^*) . Since the dependence of $f_\mu(x)$ on μ is very simple, you will not need to use the formula for Taylor series for a function of two variables – it will be enough to use the Taylor expansion of the function $\tan x$ about an appropriately chosen point. For your convenience, here are the Taylor expansions of $\tan x$ about 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ – use the one that you need:

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

$$\tan x = \frac{1}{\sqrt{3}} + \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{4}{3\sqrt{3}}\left(x - \frac{\pi}{6}\right)^2 + \frac{8}{9}\left(x - \frac{\pi}{6}\right)^3 + \frac{4}{3\sqrt{3}}\left(x - \frac{\pi}{6}\right)^4 + \dots$$

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + \dots$$

$$\tan x = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + 4\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{3}\right)^3 + \frac{44}{\sqrt{3}}\left(x - \frac{\pi}{3}\right)^4 + \dots$$

- (e) Take the lowest-order terms that contain μ and x in the Taylor expansion you obtained in part (d), and use them to find an approximate expression for the fixed points as functions of the difference $(\mu - \mu_c)$, for $\mu > \mu_c$. You will obtain that

$$x_{1,2}^* \approx \frac{\pi}{4} \pm \sqrt{\frac{1}{2}(\mu - \mu_c)} ;$$

I want to see your detailed calculations.

- (f) For $\mu > \mu_c$, determine which of the fixed points x_1^* and x_2^* is attracting and which one is repelling. You may use a graph or a calculation (using the graph is easier).
- (g) Sketch the bifurcation diagram for the 1-parameter family of ODEs (3), near the point (x_c^*, μ_c) .

Problem 4. [Bifurcation in a logistic equation with linear harvesting]

Consider a population $X(T)$ that changes according to the logistic equation and in addition is subjected to a linear harvesting, i.e., in each time interval, a part of the population is removed by harvesting, and the harvesting is assumed to be a linear function of the population at that moment. If R is the reproduction rate of the population, K is the carrying capacity, and A and B are parameters that define the harvesting, the ordinary differential equation governing the evolution of the population is

$$\frac{dX}{dT} = RX \left(1 - \frac{X}{K}\right) - (A + BX) . \quad (4)$$

- (a) It looks like the equation (4) has four parameters, but in fact two of them can be eliminated by a change of variables. Change the independent variable T and the dependent variable X to the new “time” t and “population” x by

$$t = RT , \quad x = \frac{X}{K} ,$$

and show that the four-parameter family (4) becomes the two-parameter family

$$\frac{dx}{dt} = x(1-x) - (a+bx) . \quad (5)$$

Express the new parameters, a and b , in terms of the old parameters R , K , A , and B .

- (b) Rewrite the condition $x(1-x) - (a+bx) = 0$ for a FP of (5) in the form $f(x) = g_{a,b}(x)$ with $f(x) = x(1-x)$ and $g_{a,b}(x) = a+bx$. Plot the graphs of $f(x)$ and $g_{a,b}(x)$ together, for three cases: when (5) has no FP, when (5) has exactly one FP, and when (5) has two FPs.
- (c) Write down the conditions for the equation (5) to have exactly one FP. Solve them to obtain a relation between the parameters a and b .
Hint: Recall that the graphs of $f(x)$ and $g_{a,b}(x)$ must “touch” at a point, which gives you two conditions.
- (d) Plot the relation obtained in part (c) in the (a, b) plane, and indicate how many FPs of (5) are there in each region in your plot.