MATH 4163 **Homework 2** **Due Thu, 02/01/2018**

Problem 1. [Classification of linear second order PDEs]

Consider the linear non-homogeneous second order PDE

$$y^2 u_{xx} - 2y u_{xy} + u_{yy} - u_x = 6y \,. \tag{1}$$

(a) Write the coefficients of the second derivatives,

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = y^2 u_{xx} - 2y u_{xy} + u_{yy}$$

as a symmetric 2×2 matrix

$$A(x,y) = \begin{bmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{bmatrix}$$

and determine whether the PDE is parabolic, elliptic, or hyperbolic.

(b) Consider the change of variables from (x,y) to $(\widetilde{x},\widetilde{y})$ defined by

$$\widetilde{x} = x + \frac{y^2}{2} \,, \qquad \widetilde{y} = y \,. \tag{2}$$

For your convenience, here is a table of the derivatives that will be useful later:

$$\widetilde{x}_x = 1 \,, \quad \widetilde{x}_y = y \,, \quad \widetilde{y}_x = 0 \,, \quad \widetilde{y}_y = 1 \,.$$

Compute u_x and u_{xx} in terms of the tilded function $\widetilde{u}$ and its derivatives (with respect to the tilded variables $\widetilde{x}$ and $\widetilde{y}$). Here are some instructive calculations:

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\widetilde{u}\big(\widetilde{X}(x,y),\widetilde{Y}(x,y)\big) = \widetilde{u}_{\widetilde{x}}\,\widetilde{x}_y + \widetilde{u}_{\widetilde{y}}\,\widetilde{y}_y = y\widetilde{u}_{\widetilde{x}} + \widetilde{u}_{\widetilde{y}} \,,$$

$$u_{xy} = \frac{\partial}{\partial x}u_y = \frac{\partial}{\partial x}\left(y\widetilde{u}_{\widetilde{x}} + \widetilde{u}_{\widetilde{y}}\right) = y\left(\widetilde{u}_{\widetilde{x}\widetilde{x}}\,\widetilde{x}_x + \widetilde{u}_{\widetilde{x}\widetilde{y}}\,\widetilde{y}_x\right) + \left(\widetilde{u}_{\widetilde{y}\widetilde{x}}\,\widetilde{x}_x + \widetilde{u}_{\widetilde{y}\widetilde{y}}\,\widetilde{y}_x\right) = y\widetilde{u}_{\widetilde{x}\widetilde{x}} + \widetilde{u}_{\widetilde{x}\widetilde{y}} \,.$$

(c) Compute u_{yy} in terms of the tilded function $\widetilde{u}$ and its derivatives (with respect to the tilded variables $\widetilde{x}$ and $\widetilde{y}$).

(d) Use your results from parts (a) and (b) to transform the PDE (1) to the form

$$\widetilde{u}_{\widetilde{y}\widetilde{y}}(\widetilde{x},\widetilde{y}) = 6\widetilde{y} \,. \tag{3}$$

(e) Find the general solution $\widetilde{u}(\widetilde{x},\widetilde{y})$ of the PDE (3) (by treating it as an ODE). Your answer should contain two arbitrary functions (each depending on one variable).

(f) Find the general solution $u(x,y)$ of the original PDE (1).

Problem 2. [Cauchy-Riemann equations, harmonic conjugate]

The system of first order PDEs for the functions $u(x,y)$ and $v(x,y)$

$$\begin{aligned} u_x &= v_y \,, \\ u_y &= -v_x, \end{aligned} \tag{4}$$

called *Cauchy-Riemann equations*, is of fundamental importance in complex analysis.

(a) Prove that the functions $u(x,y)$ and $v(x,y)$ satisfying (4) are *harmonic*, i.e., that each of them satisfies Laplace's equation: $\Delta u = 0$, $\Delta v = 0$.

(b) If the functions $u(x,y)$ and $v(x,y)$ satisfy (4), then the function $v(x,y)$ is said to be *harmonic conjugate* to the function $u(x,y)$. Show that the function

$$u(x,y) = 2x - x^3 + 3xy^2 \,. \tag{5}$$

is harmonic.

(c) In this and the following part of the problem you will find a harmonic conjugate $v(x,y)$ of the function $u(x,y)$ from (5). To this end, first write, say, the second equation from (4),

$$v_x = -u_y$$

(where the right-hand side is known), and integrate it with respect to x; your result for $v(x,y)$ will contain one arbitrary function of one variable.

(d) Impose the first condition from (4), namely,

$$v_y = u_x \,,$$

onto $v(x,y)$ from part (c) in order to determine the arbitrary function. Your final result for $v(x,y)$ will contain only one arbitrary constant, but no arbitrary functions.

Problem 3. [A simple constant coefficients first order PDE]

Follow the method from Sec. 2.1 of Bleecker and Csordas's book to find the constant coefficient first order PDE

$$u_t - c u_x = 0 \,, \qquad u = u(x,t) \,, \quad c = \text{const} > 0 \,.$$

The family of functions you will obtain describe waves moving to the left with speed c; this type of solutions are called *traveling waves*.

Problem 4. [Constant coefficients first order PDEs, additional conditions]

In this problem you will find the general solution of the first order PDE

$$u_x + 2u_y - 4u = \mathrm{e}^{x+y} \,, \qquad u = u(x,y) \,, \tag{6}$$

and two particular solutions corresponding to two different additional conditions. The text on pages 58–61 of Bleecker and Csordas's book will be useful.

(a) Following Sec. 2.1 of the book, consider a change of variables from (x, y) to $(\widetilde{x}, \widetilde{y})$ given by

$$\widetilde{x} = 2x - y$$
$$\widetilde{y} = x$$

(you have to understand the logic behind making this change – read pages 58 and 59; the choice of $\widetilde{y}$ is highly arbitrary). Write down the inverse change of variables (i.e., express (x, y) in terms of $(\widetilde{x}, \widetilde{y})$).

(b) Compute u_x and u_y in terms of the new function $\widetilde{u}$ and its derivatives.

(c) Plug the expressions for u_x and u_y into the PDE (6) and show that $\widetilde{u}(\widetilde{x}, \widetilde{y})$ satisfies the PDE

$$\widetilde{u}_{\widetilde{y}}(\widetilde{x}, \widetilde{y}) - 4\widetilde{u}(\widetilde{x}, \widetilde{y}) = \mathrm{e}^{-\widetilde{x}+3\widetilde{y}} . \tag{7}$$

(d) Find the general solution $\widetilde{u}(\widetilde{x}, \widetilde{y})$ of the PDE (7), which can be solved as a first order linear ODE (treating $\widetilde{x}$ as frozen) (introducing integrating factor – see pages 4 and 5 of the book).

(e) Change variables in the function $\widetilde{u}(\widetilde{x}, \widetilde{y})$ obtained in part (d) to show that the general solution of the original PDE (6) is (with φ being an arbitrary function of one variable)

$$u(x, y) = -\mathrm{e}^{x+y} + \mathrm{e}^{4x}\varphi(2x - y) . \tag{8}$$

(f) Compute the derivatives u_x and u_y of the function $u(x, y)$ given by (8), and check that $u(x, y)$ satisfies the PDE (6).

(g) Impose the additional condition

$$u(x, 0) = \sin x \tag{9}$$

on the values of the general solution $u(x, y)$ from (8) on the x-axis to find the function $u(x, y)$ satisfying both (6) and (9).

(h) Impose the additional condition

$$u(x, -x) = x^5 \tag{10}$$

on the values of the general solution $u(x, y)$ from (8) on the anti-diagonal $\{y = -x\}$ to find the function $u(x, y)$ satisfying both (6) and (10).

Problem 5. [Differential operators and integral theorems in Calculus]

In this (very easy) problem you will need some of the important results in multivariable Calculus – see, e.g., Chapter 16 in the (7th or 8th edition of) Stewart's *Calculus*.

(a) Let Σ be an oriented surface in $\mathbb{R}^3$, and $C = \partial\Sigma$ be its boundary (which is a closed line, oriented appropriately). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar-valued function of three variables. Use the Fundamental Theorem for Line Integrals to find the value of the line integral $\oint_C \nabla f(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$ (the circle on the integral simply reminds us that the curve C is closed).

(b) Recall the Stokes Theorem,

$$\oint_{\partial\Sigma} \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} = \iint_{\Sigma} \operatorname{curl} \mathbf{F}(\mathbf{x}) \cdot \mathrm{d}\mathbf{S} \,.$$

Use the Stokes Theorem and one fundamental fact about the differential operators curl, div, and grad, to compute the value of the linear integral $\oint_C \nabla f(\mathbf{x}) \cdot \mathrm{d}\mathbf{x}$ (which you have already found in part (a) by using a different method).

(c) Now let E be a 3-dimensional domain in $\mathbb{R}^3$ (think, e.g., of an ellipsoid), and let ∂E be the its boundary, i.e., a closed 2-dimensional surface, with the unit normal pointing outwards. Let $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field on $\mathbb{R}^3$. Use the Divergence Theorem and one fundamental fact about the differential operators curl, div, and grad, to show that the *flux* of the vector field $\operatorname{curl} \mathbf{G}$ through the surface of E is zero, i.e., that the surface integral $\oiint_{\partial E} \operatorname{curl} \mathbf{G}(\mathbf{x}) \cdot \mathrm{d}\mathbf{S}$ is zero. (The circle on the double integral simply reminds us that the surface ∂E is closed.)