MATH 4163 Homework 2 Due Thu, Sep 6, 2012

Problem 1. Solve the following partial differential equation:

$$u_{xyy}(x, y) = 4\mathrm{e}^x \sin \frac{y}{2} \; .$$

Check by direct substitution that the general solutions you found satisfies the PDE.

Hint: Recall that the general solution of a PDE of order n for a function of d variables contains exactly n arbitrary functions, each of which is a function of $(d-1)$ variables.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be an even function and $g : \mathbb{R} \to \mathbb{R}$ be an odd function, and a be an arbitrary positive number.

(a) Show that $\int_{-a}^{0} f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x$, and $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$.

(b) Show that $\int_{-a}^{0} g(x)\,\mathrm{d}x = -\int_0^a g(x)\,\mathrm{d}x$, and $\int_{-a}^{a} g(x)\,\mathrm{d}x = 0$.

(c) Are the functions that $f \cdot f$, $f \cdot g$, and $g \cdot g$ even, odd, or neither even nor odd? Prove your claims. Here $f \cdot g$ is defined as $(f \cdot g)(x) := f(x)\,g(x)$, and similarly for the other products.

(d) Each function $h : \mathbb{R} \to \mathbb{R}$ can be written in a unique way as a sum of an even and an odd function:

$$h(x) = h_{\mathrm{e}}(x) + h_{\mathrm{o}}(x) \; .$$

Express h_{e} and h_{o} in terms of the function h.

(e) Find h_{e} and h_{o} if $h(x) = 3 + 7x^2 + \sin^3 x - e^{2x}$.

Problem 3.

(a) For any real θ, use Euler's formula, $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$, in order to express $\cos\theta$ and $\sin\theta$ in terms of $\mathrm{e}^{i\theta}$ and $\mathrm{e}^{-i\theta}$.

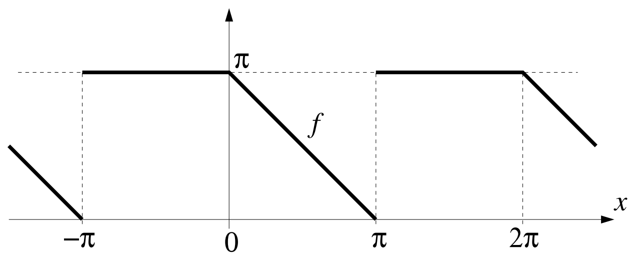
(b) Let the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ have Fourier series $f(x) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{inx}$. We proved in class that, for a real-valued function f, the following relation holds: $c_{-n} = c_n^*$ for any $n \in \mathbb{Z}$. If $c_n = \alpha_n + i\beta_n$, where $\alpha_n = \mathrm{Re}\, c_n \in \mathbb{R}$ and $\beta_n = \mathrm{Im}\, c_n \in \mathbb{R}$, find $\mathrm{Re}\,(c_n \mathrm{e}^{inx})$ and $\mathrm{Im}\,(c_n \mathrm{e}^{inx})$ in terms of α_n, β_n, $\cos nx$, and $\sin nx$.

(c) Rewrite the FS of f from part (b) in the form $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, and express a_n and b_n in terms of c_n.

Problem 4. Let f be a periodic function of period 2π which for x between $-\pi$ and π is defined as

$$f(x) = \begin{cases} \pi \ , & -\pi < x \le 0 \ , \\ \pi - x \ , & 0 < x \le \pi \ ; \end{cases}$$

the graph of f is sketched in the figure below.



The Fourier series of f is the following (you do not have to prove this!):

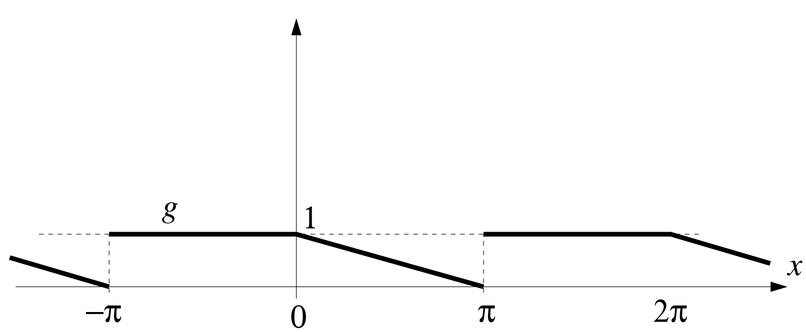
$$\begin{aligned} f(x) &= \frac{3\pi}{4} + \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right) \\ &\quad - \sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots \ . \end{aligned} \tag{1}$$

(a) Use (1) to prove the identity $\displaystyle\sum_{\text{odd positive } n} \frac{1}{n^2} = \frac{\pi^2}{8}$.

(b) What identity do you obtain if you set $x = \frac{\pi}{2}$ in (1)?

(c) If you plug $x = \pi$ in the Fourier series in the right-hand side of (1), will the Fourier series converge (just say "yes" or "no")? If it converges, to what value does it converge? (Do not attempt to compute the value of the infinite sum, just write a couple of sentences about what the general theory predicts.)

(d) Let the function g be a periodic function of period 2π sketched in the figure below. Using (1), find the Fourier series of the function g.

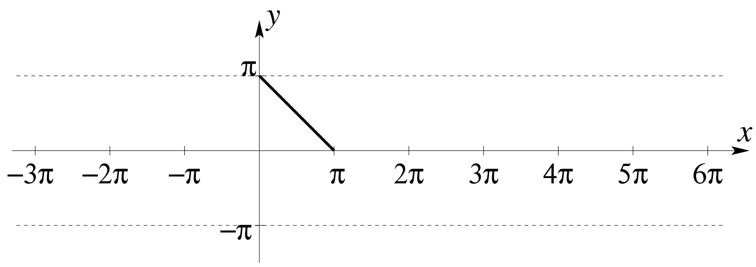


Problem 5.

(a) Extend the function

$$f(x) = \pi - x \quad \text{for } x \in (0, \pi)$$

as an *even* periodic function $f_{\text{even}}(x)$ of period 2π. Draw the graph of $f_{\text{even}}(x)$.



(b) *Without doing any computations,* from the expressions below choose the one that gives the Fourier series of the function $f_{\text{even}}(x)$ obtained in part (a).

$$f_{\text{even}}(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos nx$$

$$f_{\text{even}}(x) = -\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos nx$$

$$f_{\text{even}}(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \sin nx$$

$$f_{\text{even}}(x) = -\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \sin nx$$

$$f_{\text{even}}(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n^2} \cos nx - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx$$

Give the reasons for your choice – a couple of sentences only, but *please be concrete*!
Hint: One of the Fourier coefficients of a periodic function is related to the average value of the function.

(c) Extend the function

$$f(x) = \pi - x \quad \text{for } x \in (0, \pi)$$

as an *odd* periodic function $f_{\text{odd}}(x)$ of period 2π. Draw the graph of $f_{\text{odd}}(x)$.

(d) The Fourier series of the function $f_{\text{odd}}(x)$ obtained in part (c) is

$$f_{\text{odd}}(x) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \; .$$

Suppose that you want to extend the function $f(x) = \pi - x$ for $x \in (0, \pi)$ to the whole real line and then to find the Fourier series of the extension, with the purpose of computing the values of $f(x)$ for $x \in (0, \pi)$ numerically by approximating this value by a partial sum of the corresponding Fourier series. Which expansion will be more beneficial numerically – the one of $f_{\text{even}}(x)$ from part (b) or the one of $f_{\text{odd}}(x)$ given above – if you want to use as few as possible terms in the series in order to achieve give accuracy of your answer? Explain your choice with one sentence only.

(e) Looking at the graphs of $f_{\text{even}}(x)$ from part (b) and $f_{\text{odd}}(x)$ from part (c), can you explain the decay of the Fourier coefficients in the Fourier series for $f_{\text{even}}(x)$ and $f_{\text{odd}}(x)$ from the properties of these functions (discontinuous, continuous, differentiable, etc.)?