MATH 4073 Homework 2 Due Fri, 09/10/10

Problem 1. Find the limits and the convergence rates as $n \to \infty$ of the following sequences (by using the Taylor expansions of the functions):

(a) $\lim\limits_{n\to\infty} \sin\left(\frac{1}{n^2}\right)$;

(b) $\lim\limits_{n\to\infty} \left(\sin\frac{1}{n}\right)^2$;

(c) $\lim\limits_{n\to\infty} \dfrac{n^2}{n^5+7n}$;

(d) $\lim\limits_{n\to\infty} \left[(n+5)^{1/3} - n^{1/3}\right]$.

Hints: (c) You can rewrite the expression as $\frac{n^2}{n^5+7n} = \frac{n^2}{n^5}\frac{1}{1+\frac{7}{n^5}} = \frac{1}{n^3}\frac{1}{1-\left(-\frac{7}{n^5}\right)}$, and use the formula for the sum of a geometric series, $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ to expand the second factor.

(d) Write $(n+5)^{1/3} - n^{1/3} = n^{1/3}\left[\left(1+\frac{5}{n}\right)^{1/3} - 1\right]$, and use that the Taylor expansion of $(1+x)^\alpha$ around $x = 0$ when α is not equal to a positive integer is given by

$$(1+x)^\alpha = 1 + \tfrac{\alpha}{1!}x + \tfrac{\alpha(\alpha-1)}{2!}x^2 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots .$$

Problem 2. In the limit $h \to 0$, find the real numbers P, Q, and R, and the integers p, q, and r in the following relations (by using the Taylor expansions of the corresponding functions):

(a) $\dfrac{e^h - \cos h}{h} = P + \mathcal{O}(h^q)$;

(b) $\cos(\sin h) = Q + \mathcal{O}\left(h^p\right)$;

(c) $\ln\sqrt{3+h} = R + \mathcal{O}(h^r)$.

Problem 3. Let the sequence p_n satisfy the recurrence relation

$$p_n = \frac{1}{3}\,p_{n-1} + \frac{1}{12}\,p_{n-2} \qquad \text{for} \;\; n \geq 2 ,$$

and the initial conditions $p_0 = 2$, $p_1 = \frac{1}{3}$.

(a) Find the general solution of the recurrence relation (without imposing the initial conditions).

(b) Find the solution of the recurrence relation with the initial conditions.

(c) Think of a sequence $\{\beta_n\}_{n=1}^{\infty}$ such that $p_n = \mathcal{O}(\beta_n)$.

(d) Do you think that solving this recurrence relation numerically can get us into trouble (like what happens with the recurrence relation from Example 3 from Section 1.3 of the book)? Explain why or why not.

Problem 4.

(a) A very convenient test for convergence or divergence of a series is the so-called *integral test* (see, e.g., J. Stewart, *Calculus*, 5th or 6th edition, Sec 12.3). It says that, if f is a continuous, positive and decreasing function on $[1, \infty)$, and if $a_n = f(n)$, then the integral $\int_1^\infty f(x)\,\mathrm{d}x$ is convergent if and only if the infinite sum $\sum_{n=1}^{\infty} a_n$ is convergent.

This idea can also be used to estimate the error in truncating the infinite series $\sum_{n=1}^{\infty} a_n$ (where the terms $a_n = f(n)$ for a function f satisfying the above properties). Namely, show that the truncation error E_M satisfies

$$E_M := \left| \sum_{n=M+1}^{\infty} a_n \right| \le \int_M^\infty f(x)\,\mathrm{d}x \ .$$

It is enough to draw a nice picture and a couple of words of explanation.

Hint: Interpret the integral as an area, and the sum as a sum of the areas of rectangles of height $a_n = f(n)$ and width 1.

(b) Use the property proved in (a) to estimate the number M of terms needed to compute the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ with error not exceeding 10^{-3}.

(c) Now you will use Mathematica to check that your estimate from part (b) indeed works. The exact value of the infinite sum is $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232323371113819151\ldots$.

To compute $\displaystyle\sum_{n=1}^{M} \frac{1}{n^4}$ using Mathematica, one can open Mathematica and type

```
Sum[1/n^4, {n,1,M}]
```

where `{n,1,M}` tells Mathematica that the summation is over all integers from 1 to `M` (where `M` is the value found in part (b)). Having typed this, press the SHIFT button, hold it down, and press RETURN. The output will be a ratio of two gigantic integers. To obtain the numerical value of this ratio with, say, 30 digits of accuracy, type

```
N[Sum[1/n^4, {n,1,M}], 30]
```

and again press RETURN while holding down SHIFT. Find the true difference of the exact value of the infinite sum and the value of its truncation, and compare this difference with the rigorous error bound found in part (b). Discuss briefly.

Problem 5. On August 2, 2010, Shigeru Kondo used Alexander Yee announced that they have calculated 5,000,000,000,000 digits of π. They used the a program called y-cruncher, developed by Yee, and performed their computations on a single desktop computer built by Kondo; the computation took 90 days (between 6:19 p.m. on May 4 and 1:21 p.m. on August 3, 2010). You can see their announcement and details on their work at

`http://www.numberworld.org/misc_runs/pi-5t/announce_en.html`

`http://www.numberworld.org/misc_runs/pi-5t/details.html`

In their computations Kondo and Yee used the following formula derived by the brothers David and Gregory Chudnovsky, who relied on some ideas of the famous Inidian mathematician Srinivasa Ramanujan (1887–1920):

$$\frac{1}{\pi} = \frac{\sqrt{10005}}{4270934400} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(k!)^3\,(3k)!} \, \frac{13591409 + 545140134k}{640320^{3k}} \,.$$

In this problem you will use Mathematica to find the rate of convergence of the right-hand side of this formula to the exact value of $\frac{1}{\pi}$. You can define the function `chud[n]` which computes the sum of the first `n` terms of Chudnovsky's formula:

```
termPi[k_]=(-1)^k*(6*k)!/(k!)^3/(3*k)!*(13591409+545140134*k)/640320^(3*k)

chud[n_]=Sqrt[10005]/4270934400*Sum[termPi[k], {k, 0, n}]
```

(again, after you type each line, press SHIFT, hold it down, and press RETURN). The underscores after `k` and `n` in `termPi[k_]` and `chud[n_]` tell Mathematica that we are defining new functions, and `k` and `n` the variables of these functions.

To find the numerical value with accuracy of 1000 digits of the difference between the exact value of $\dfrac{1}{\pi}$ and the partial sum of the sum containing, say, 8 terms – which in our notations will be equal to `chud[7]` – you can type the following:

```
N[chud[7] - 1/Pi, 1000]
```

There will a problem, however, and Mathematica will complain that its internal precision limit is not enough for the computation (try it!). That is why you have to type

```
Block[{$MaxExtraPrecision = 1000}, N[chud[7] - 1/Pi, 1000]]
```

(a) Compute the numerical values of the absolute error $E_n = \left|\frac{1}{\pi} - \texttt{chud}[n]\right|$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$, and write your results in a table (there is no need to write more than 3–4 digits of accuracy of E_n in the table).

(b) For the values of n used in part (a), show that your numerical results give $\frac{E_{n+1}}{E_n} \approx 10^{-14}$. Can you express E_n approximately in terms of E_0? (Nothing sophisticated, just an approximate formula like the ones in Definition 1.17, Section 1.3 of the book.)

(c) If you write the sequence $\{E_n\}_{n=0}^{\infty}$ of the errors defined in part (a) in the form $E_n = L + \mathcal{O}(\beta_n)$, then what are the limit L and the sequence $\{\beta_n\}_{n=0}^{\infty}$?