MATH 4073 Homework 2 Due Thu, 09/10/09

Problem 1. Find the limits and the convergence rates as $n \to \infty$ of the following sequences (by using the Taylor expansions of the functions):

(a) $\lim_{n\to\infty} \left(\frac{1}{n^3} - \sin \frac{1}{n^3} \right)$;

(b) $\lim_{n\to\infty} \frac{n-1}{n^3+2}$;

(c) $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n} \right)$.

Hint: Write $\sqrt{n+1} - \sqrt{n} = \sqrt{n}\left(\sqrt{1+\frac{1}{n}} - 1\right)$, and use that the Taylor expansion of $(1+x)^\alpha$ around $x=0$ when α is not equal to a positive integer is given by

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots .$$

Problem 2. In the limit $h \to 0$, find the real numbers P, Q, and R, and the integers p, q, and r in the following relations (by using the Taylor expansions of the corresponding functions):

(a) $\frac{e^h - 1}{h} = P + \mathcal{O}\left(h^p\right)$;

(b) $\frac{e^h - \cos h - h}{h^2} = Q + \mathcal{O}(h^q)$;

(c) $\frac{\ln(1 + 3\sin^5 h)}{h^5} = R + \mathcal{O}(h^r)$.

Problem 3. Let the sequence p_n satisfy the recurrence relation

$$p_n = \frac{5}{6}\,p_{n-1} - \frac{1}{6}\,p_{n-2} \qquad \text{for} \quad n \geq 2 ,$$

and the initial conditions $p_0 = 2$, $p_1 = \frac{5}{6}$.

(a) Find the general solution of the recurrence relation (without imposing the initial conditions).

(b) Find the solution of the recurrence relation with the initial conditions.

(c) Think of a sequence $\{\beta_n\}_{n=1}^{\infty}$ such that $p_n = \mathcal{O}(\beta_n)$.

(d) Do you think that solving this recurrence relation numerically can get us into trouble (like what happens with the recurrence relation from Example 3 from Section 1.3 of the book)? Explain briefly.

Problem 4.

(a) A very convenient test for convergence or divergence of a series is the so-called *integral test* (see, e.g., J. Stewart, *Calculus*, 5th or 6th edition, Sec 12.3). It says that, if f is a continuous, positive and decreasing function on $[1, \infty)$, and if $a_n = f(n)$, then the integral $\int_1^\infty f(x)\,\mathrm{d}x$ is convergent if and only if the infinite sum $\sum_{n=1}^\infty a_n$ is convergent.

This idea can also be used to estimate the error in truncating the infinite series $\sum_{n=1}^\infty a_n$ (where the terms $a_n = f(n)$ for a function f satisfying the above properties). Namely, show that the truncation error E_M satisfies

$$E_M := \left| \sum_{n=M+1}^{\infty} a_n \right| \leq \int_M^\infty f(x)\,\mathrm{d}x \ .$$

It is enough to draw a nice picture and a couple of words of explanation.

Hint: Interpret the integral as an area, and the sum as a sum of the areas of rectangles of height $a_n = f(n)$ and width 1.

(b) Use the property proved in (a) to estimate the number M of terms needed to compute the sum of the series $\sum_{n=1}^\infty \frac{1}{n^2}$ with error not exceeding 10^{-3}.

(c) Now you will use Mathematica to check that your estimate from part (b) indeed works. The exact value of the infinite sum is $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6} = 1.6449340668482264364 72\ldots$.

To compute $\displaystyle\sum_{n=1}^{M} \frac{1}{n^2}$ using Mathematica, one can open Mathematica and type

```
Sum[1/n^2, {n, M}]
```

where `M` is the value found in part (b). Having typed this, press the SHIFT button, hold it down, and press RETURN. The output will be a ratio of two gigantic integers. To obtain the numerical value of this ratio with, say, 30 digits of accuracy, type

```
N[Sum[1/n^2, {n, M}],30]
```

and again press RETURN while holding down SHIFT. Find the true difference of the exact value of the infinite sum and the value of its truncation, and compare this difference with the rigorous error bound found in part (b). Discuss briefly.

Problem 5. This is another problem in which you will use Mathematica. This time you will find the rate of convergence of a very efficient method for computing the value of π based on the following formula proved by the famous Inidian mathematician Srinivasa Ramanujan (1887–1920):

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + 26390n)}{(n!)^4\,396^{4n}}$$

Use Mathematica to find the error in approximating π by the inverse of the right-hand side of the formula above. You can define the function `termPi` such that if the argument of this function is n, it gives the value of the nth term in the sum above as follows:

```
termPi[n_] := Sqrt[8]/9801*(4*n)!*(1103+26390*n)/(n!)^4/396^(4*n)
```

(again, after you type all this, press SHIFT, hold it down, and press RETURN). The underscore after `n` in `termPi[n_]` tells Mathematica that this is a definition of a function, and `n` is the variable.

To find the numerical value of the difference between the exact value of $\dfrac{1}{\pi}$ and the partial sum $\sum_{n=0}^{\mathtt{M}} \mathtt{termPi}[n]$ of the Ramanujan's series with 100 digits of accuracy, you can type

```
N[ Pi - 1/Sum[termPi[n], {n, 0, M}], 100]
```

Note that, since this time the summation starts at 0, you typed `{n,0,M}`; when the summation starts from 1, then one can type either `{n,1,M}` or `{n,M}`.

(a) Compute the numerical values of the absolute error

$$E_M = \left| \pi - \left(\sum_{n=0}^{M} \mathtt{termPi}[n] \right)^{-1} \right|$$

for $M = 0, 1, 2, 3, 4, 5, 6$, and write your results in a table (there is no need to write more than 3–4 digits of accuracy of E_M in the table).

(b) For the values of M used in part (a), show that your numerical results give $\frac{E_{M+1}}{E_M} \approx 10^{-8}$. Can you express E_M approximately in terms of E_0? (Nothing sophisticated, just an approximate formula like the ones in Definition 1.17, Section 1.3 of the book.)

(c) If you write the sequence $\{E_M\}_{M=0}^{\infty}$ of the errors defined in part (a) in the form $E_M = L + \mathcal{O}(\beta_M)$, then what are the limit L and the sequence $\{\beta_M\}_{M=0}^{\infty}$?