MATH 3423 Homework 2 Due Thu, 09/05/19

Problem 1. Assume that the physical system described by some action is *autonomous*, which means that its Lagrangian does not depend explicitly on time, i.e., if $L = L(q(t), \dot{q}(t))$ (as opposed to the general case $L = L(q(t), \dot{q}(t), t)$). Let $\widetilde{q}(t)$ be a solution of the corresponding Euler-Lagrange equation. Define the *energy* as

$$E := \dot{q}\,\frac{\partial L}{\partial \dot{q}} - L\ , \tag{1}$$

or, more precisely, the energy is a function of time defined in (1), where the arguments $q(t)$ and $\dot{q}(t)$ are replaced by the true functions $\widetilde{q}(t)$ and $\dot{\widetilde{q}}(t)$, where $\widetilde{q}(t)$ is a solution of the Euler-Lagrange equation:

$$E(t) := \dot{\widetilde{q}}(t)\,\frac{\partial L}{\partial \dot{q}}\big(\widetilde{q}(t), \dot{\widetilde{q}}(t)\big) - L\big(\widetilde{q}(t), \dot{\widetilde{q}}(t)\big)\ .$$

In this problem you will prove that the energy of an autonomous system does not depend on time: $E = \text{const}$.

(a) The total derivative of $L = L(q(t), \dot{q}(t), t)$ with respect to t is

$$\frac{\mathrm{d}}{\mathrm{d}t} L(q(t), \dot{q}(t), t) = \frac{\partial L}{\partial q}\,\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}}\,\frac{\mathrm{d}\dot{q}}{\mathrm{d}t} + \frac{\partial L}{\partial t} = \frac{\partial L}{\partial q}\,\dot{q} + \frac{\partial L}{\partial \dot{q}}\,\ddot{q} + \frac{\partial L}{\partial t}\ .$$

How does this expression change when L does not depend explicitly on time, i.e., $L = L(q(t), \dot{q}(t))$?

(b) Take the time derivative of $\dot{q}\,\dfrac{\partial L}{\partial \dot{q}}$ and use the Euler-Lagrange equation to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right) = \dot{q}\,\frac{\partial L}{\partial q} + \ddot{q}\,\frac{\partial L}{\partial \dot{q}}\ .$$

Here and in the next part of the problem you may skip the tilde sign to simplify the notations.

(c) From your results in parts (a) and (b) derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\,\frac{\partial L}{\partial \dot{q}} - L\right) = 0\ .$$

What does this relation imply about the energy (1) of the system?

Problem 2.

(a) Write down and solve the Euler-Lagrange equation for $L(q, \dot{q}, t) = \dot{q}^2 + 2q\dot{q} - 16q^2$. (The Euler-Lagrange equation will be an equation that you know very well.)

(b) Write down and solve the Euler-Lagrange equation for $L(q, \dot{q}, t) = t\dot{q} + \dot{q}^2$. (The general solution will be a quadratic polynomial of the time.)

Problem 3. Sometimes the energy conservation can be used to simplify the calculations, i.e., one can write down and solve the energy conservation instead of solving the Euler-Lagrange equation. This is what you will do in this problem.

(a) Consider the action functional

$$I[q] = \int_{t_1}^{t_2} \frac{1 + q^2}{\dot{q}^2} \, \mathrm{d}t \; . \tag{2}$$

Clearly, the system is autonomous (what does this mean?), so that its energy E (defined in (1)) must be conserved (i.e., constant). For the action (2), write down the expression for its energy E; you already know that E does not depend on time, so that you will treat it as an arbitrary constant.

(b) Find the general solution of the first order differential equation derived in part (a). It will contain two arbitrary constants – the energy E and one constant coming from the integration. You may use that $\displaystyle\int \frac{\mathrm{d}q}{\sqrt{1+q^2}} = \sinh^{-1} q$, where $\sinh^{-1}$ is the inverse function of sinh.

Problem 4.

(a) Consider the system described by the Lagrangian

$$L(q, \dot{q}, t) = \dot{q}^2 - q^2 + 2q \sin t \; . \tag{3}$$

Write down the Euler-Lagrange equation of this system. (The equation you will obtain is the equation describing a periodically forced harmonic oscillator.)

(b) Find the general solution of the Euler-Lagrange equation derived in part (a). The Euler-Lagrange equation in this problem is a linear non-homogeneous ODE with constant coefficients; if you have forgotten how to solve such equations (in particular, how to look for a particular solution of the ODE), look at the Appendix at the end of this homework.

(c) The system described by the Lagrangian (3) is clearly not autonomous, so its energy $E(t)$ defined by (1) will depend on time. Use (1) to show that the energy E can be written as $E = \dot{q}^2 + q^2 - 2q\sin t$.

(d) Take time derivative of the energy obtained in part (c), and use that along a true trajectory $q(t)$, the Euler-Lagrange equation (derived in part (a)) holds, in order to obtain that the energy changes with time as $\dfrac{\mathrm{d}E}{\mathrm{d}t} = -2q(t)\cos t$.

Problem 5. Consider a loop made of metal wire; let the loop be in the (x, y)-plane (i.e., $z = 0$) in the 3-dimensional space. Consider a membrane (or, equivalently, a soap film) whose end is attached to the wire. If there were no gravity, the equilibrium position of the membrane will be in the plane $z = 0$ because of the surface tension. We will formulate the problem as a minimization problem of an action functional.

Let the domain in the (x, y)-plane that is surrounded by the wire be denoted by D. Let the function $u(x, y, t)$ describe the position of the membrane at time t, i.e., let the equation of the membrane at time t be

$$z = u(x, y, t) \; .$$

Define the following notations:

- ρ is the area density of the mass of the membrane (unit $\mathrm{kg/m^2}$),
- τ is the surface tension (unit $\mathrm{N/m} = \mathrm{kg/s^2}$),
- $f(x, y, t)$ is the area density of the external forces, i.e., force per unit area of the membrane; for example, the area density of gravity force is $f = -\rho g$ (the minus sign reflects the fact the that gravity acceleration $\mathbf{g}$ points downward).

In the case when the unknown function u depends on more than one variable, the situation is the following. Let u be a function depending on time t and on the spatial coordinate(s) $\mathbf{r}$; in this particular problem $\mathbf{r}$ stands for $(x, y) \in \mathbb{R}^2$. The action for the function $u(t, \mathbf{r})$ is given by

$$I[u] = \int_{t_1}^{t_2} \iint_D \mathscr{L}\big(u, \nabla u, u_t, \mathbf{r}, t\big)\, \mathrm{d}A\, \mathrm{d}t \; ,$$

where D is the given domain in (x, y)-plane (surrounded by the wire), and $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ is the area element in the (x, y)-plane. The function

$$\mathscr{L}\big(u, \nabla u, u_t, \mathbf{r}, t\big) := \mathscr{L}\big(u, u_x, u_y, u_t, x, y, t\big)$$

is called *Lagrangian density*. The Euler-Lagrange equation in this case is

$$\frac{\partial \mathscr{L}}{\partial u} - \nabla \cdot \left(\frac{\partial \mathscr{L}}{\partial \nabla u} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial u_t} \right) = 0 \; ,$$

i.e.,

$$\frac{\partial \mathscr{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathscr{L}}{\partial u_y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial u_t}\right) = 0 \ .$$

The Lagrangian density is equal to the difference of the area density of the kinetic energy and the area density of the potential energy. The area density of the kinetic energy is

$$\mathscr{T} = \frac{\rho u_t^2}{2} \ ;$$

the area density of the gravitational potential energy is

$$\mathscr{U}_{\text{grav}} = \rho g u \ .$$

The total potential energy due to the surface tension is equal to the surface tension times the change of the area of the membrane due to the shape of the membrane. The area of the membrane when it is flat is $A(D)$ (recall that D is the domain the (x, y)-plane surrounded by the wire), and its area at time t is given by (look into any Calculus book)

$$\iint_D \sqrt{1 + |\nabla u|^2}\,\mathrm{d}A = \iint_D \left(1 + |\nabla u|^2\right)^{1/2} \mathrm{d}x\,\mathrm{d}y \approx \iint_D \left(1 + \tfrac{1}{2}|\nabla u|^2\right) \mathrm{d}x\,\mathrm{d}y$$

where $|\nabla u|^2 = u_x^2 + u_y^2$. Here we used the fact that, and for $|\xi| < 1$ and $\alpha \notin \{0, 1, 2, 3, \ldots\}$,

$$(1 + \xi)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} \xi^k \ ,$$

where

$$\binom{\alpha}{k} := \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} \ .$$

This gives us that the total elastic potential energy (due to the surface tension) is

$$\tau \left[\iint_D \left(1 + |\nabla u|^2\right)^{1/2} \mathrm{d}x\,\mathrm{d}y - (\text{area of } D)\right] \approx \frac{\tau}{2} \iint_D \left(u_x^2 + u_y^2\right) \mathrm{d}x\,\mathrm{d}y$$

Therefore the area density of the elastic potential energy

$$\mathscr{U}_{\text{elastic}} = \frac{\tau}{2}\left(u_x^2 + u_y^2\right) \ .$$

Putting everything together, we obtain

$$\mathscr{L}\big(u, u_x, u_y, u_t, x, y, t\big) = \mathscr{T} - (\mathscr{U}_{\text{grav}} + \mathscr{U}_{\text{elastic}}) = \frac{\rho u_t^2}{2} - \rho g u - \frac{\tau}{2}\left(u_x^2 + u_y^2\right) \ . \qquad (4)$$

(a) Derive the Euler-Lagrange equation corresponding to the Lagrangian density (4). In writing your final result, use the Laplacian operator Δ defined by $\Delta u = u_{xx} + u_{yy}$. The equation you will obtain can be written in the form

$$\Delta u - \frac{1}{c^2}\, u_{tt} = \frac{\rho g}{\tau} \ ,$$

where c is the speed of the propagation of the waves in the membrane. What is c in the equation you derived?

(b) Now assume that the rim of the membrane (i.e., the wire on which the membrane is suspended) is circular with radius R. Consider the static equilibrium of the membrane, when the function u does not depend on t. Because of the symmetry of the system, it is clear that in polar coordinates (r, θ) the shape of the membrane will depend only on R, so that we can set $u(x, y, t) = U(r)$, where $r = \sqrt{x^2 + y^2}$. A tedious calculation (you do *not* need to do it here!) shows that in polar coordinates the Laplacian is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \ .$$

Use this to show that the function $U(r)$ satisfies the boundary-value problem

$$\begin{aligned} &\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\, U'(r)\right] = \frac{\rho g}{\tau} \ , \qquad r \in [0, R] \ , \\ &U(R) = 0 \ , \quad |U(r)| < \infty \end{aligned} \tag{5}$$

(the condition $|U(r)| < \infty$ is imposed for obvious physical reasons).

(c) Integrate the ODE for $U(r)$ from (5) twice to find its general solution. The condition $|U(r)| < \infty$ shows that one of the integration constants will be zero; use the condition $U(R) = 0$ to find the particular solution of (5).

(d) At which point will the membrane hang most? Give physical reasons for your answer. Write down an expression for the maximum hanging of the membrane. How does it depend on the radius R?

APPENDIX

General solution of a linear homogeneous ODE with constant coefficients:

To find the general solution of the linear homogeneous ODE with constant coefficients $Ly = 0$, where

$$Ly := a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y \ , \tag{6}$$

first solve the characteristic equation of this ODE. Each root of the characteristic equation contributes a term to the general solution of the ODE:

- each real root r of the characteristic equation of multiplicity p contributes to the general solution a term

$$P_{p-1}(x)\, e^{rx} \ ,$$

  where $P_{p-1}(x)$ is a polynomial of degree $(p-1)$;

- each conjugate pair of complex roots $\alpha \pm i\beta$ of the characteristic equation, where each of the the two roots has multiplicity p, contributes to the general solution a term

$$e^{\alpha x} \left[Q_{p-1}(x) \cos \beta x + R_{p-1}(x) \sin \beta x \right] \ ,$$

  where $Q_{p-1}(x)$ and $R_{p-1}(x)$ are polynomials of degree $(p-1)$.

Particular solutions of a linear non-homogeneous ODE with constant coefficients

The general solution of the linear non-homogeneous ODE with constant coefficients $Ly = f(x)$ (where Ly is given by the expression (6) above) is equal to the sum of the general solution of the associated homogeneous equation $Ly = 0$ and a particular solution of $Ly = f(x)$. First solve the characteristic equation of $Ly = 0$ and find the general solution of $Ly = 0$, and then find a particular solution of $Ly = f(x)$ by doing the following:

- in the case $f(x) = e^{cx} P_m(x)$, if c is a root of the characteristic equation of $Ly = 0$ with multiplicity s, then look for a particular solution $y_p(x)$ of $Ly = f(x)$ of the form

$$y_p(x) = x^s\, e^{cx}\, Q_m(x) \ ,$$

  where $Q_m(x)$ is a polynomial of degree m;

- in the case $f(x) = e^{cx} \left[P_{m_1}(x) \cos dx + \widetilde{P}_{m_2}(x) \sin dx \right]$, if $c + id$ is a root of the characteristic equation of $Ly = 0$ with multiplicity s, then define $m := \max(m_1, m_2)$, and look for a particular solution $y_p(x)$ of $Ly = f(x)$ of the form

$$y_p(x) = x^s\, e^{cx} \left[Q_m(x) \cos dx + \widetilde{Q}_m(x) \sin dx \right] \ ,$$

  where $Q_m(x)$ and $\widetilde{Q}_m(x)$ are polynomials of degree m.

If the equation has the form $Ly = f_1(x) + f_2(x)$, its general solution is a sum of the general solution of $Ly = 0$ and the particular solutions of the equations $Ly = f_1(x)$ and $Ly = f_2(x)$.