MATH 3423 Homework 2 Due Fri, 9/6/13

Problem 1. The so-called *error function* is defined as

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} \, d\xi \ .$$

(a) Suppose that you want to write an initial-value problem for a first-order ordinary equations whose solution is $\operatorname{erf}(t)$; look for initial-value problem of the form

$$\begin{aligned} \frac{dy}{dt} &= f(t, y) \ , \qquad t > 0 \\ y(0) &= y_0 \ . \end{aligned}$$

Find the explicit form of the function $f(t, y)$ and the numerical value of the constant y_0. Show your calculations and state explicitly what results you use in your derivation.

(b) Is $\operatorname{erf}(t)$ an even or an odd function (or neither)? Prove your claim.

(c) If you know that $\int_{-\infty}^{\infty} e^{-\xi^2} \, d\xi = \sqrt{\pi}$, find the limits $\lim_{z \to -\infty} \operatorname{erf}(z)$ and $\lim_{z \to \infty} \operatorname{erf}(z)$.

(d) In Lecture 3 we showed that the solution of the initial value problem for the heat equation on $\mathbb{R}$,

$$\begin{aligned} u_t(x, t) &= \alpha^2 \, u_{xx}(x, t) \ , \qquad x \in \mathbb{R} \ , \quad t > 0 \ , \\ u(x, 0) &= g(x) \end{aligned}$$

(where $g(x)$ is a function satisfying $\lim_{|x| \to \infty} f(x) = 0$), is given by

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{G}(k) \, e^{-\alpha^2 k^2 t} \, e^{ikx} \, dk \ ,$$

with $\widehat{G} = \mathcal{F}\{g\}$. Using the Convolution Theorem (equation (17) on page 851), one easily obtains equation (7) on page 857:

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} \int_{-\infty}^{\infty} g(y) \, e^{-(x-y)^2/(4\alpha^2 t)} \, dy$$

(you do *not* have to derive this here!). Write down the explicit form of the solution $u(x, t)$ for $g(x) = H(x)$, where H is the Heaviside function; express your expression for $u(x, t)$ in terms of the error function.

Problem 2. In this problem you will find the Fourier transform of the so-called *Yukawa potential*, named after the Japanese physicist Hideki Yukawa (1907–1981), recipient of the 1949 Nobel Prize for Physics for research on the theory of elementary particles. The Yukawa potential is important in plasma physics and elementary particle physics.

The Yukawa potential is given by the expression

$$u(\mathbf{r}) = \frac{\mathrm{e}^{-\alpha\rho}}{\rho}\,, \qquad \rho := |\mathbf{r}| \tag{1}$$

(we omitted some inessential overall constants). Please follow the steps below.

(a) As a preliminary calculation, show that

$$\int_0^\pi \mathrm{d}\phi\, \sin(\phi)\, \mathrm{e}^{-\mathrm{i}k\rho\cos\phi} = \frac{2\sin(k\rho)}{k\rho}$$

(where $\mathrm{i} := \sqrt{-1}$). You will need to use the relation

$$\sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}\,. \tag{2}$$

In the calculation of the integral, use what you know from Calculus, do not worry about the fact that some constants are not real numbers.

(b) Show that

$$\int_0^\infty \mathrm{d}\rho\, \mathrm{e}^{-\alpha\rho}\, \sin(k\rho) = \frac{k}{k^2+\alpha^2}\,.$$

Hint: You can either use integration by parts (in a little bit tricky way) or the following method: write $\sin(k\rho)$ in terms of $\mathrm{e}^{\pm\mathrm{i}k\rho}$ as in (2), then the integral becomes a sum of two easy integrals of exponents.

(c) In the triple Fourier transform $\widehat{U}(\mathbf{k})$ of $u(\mathbf{r})$, change the Cartesian coordinates to spherical coordinates, (ρ, θ, ϕ), by

$$x = \rho\sin\phi\cos\theta\,, \quad y = \rho\sin\phi\sin\theta\,, \quad z = \rho\cos\phi$$

(note that McQuarrie uses different notations on page 852; my notations are as in Stewart's *Calculus*). Show that if you take $\mathbf{k}$ to be in the direction of the positive z-axis (which you can do without loss of generality), then

$$\begin{aligned}
\widehat{U}(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbb{R}^3} u(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}^3\mathbf{r} \\
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty \mathrm{d}\rho\, \rho^2 \int_0^{2\pi} \mathrm{d}\theta \int_0^\pi \mathrm{d}\phi\, \sin(\phi)\, \frac{\mathrm{e}^{-\alpha\rho}}{\rho}\, \mathrm{e}^{-\mathrm{i}k\rho\cos\phi} \\
&= \sqrt{\frac{2}{\pi}}\, \frac{1}{k^2+\alpha^2}\,.
\end{aligned}$$

Please write clearly all details of your calculations.

Problem 3. Give direct proofs of the following identities:

(a) $(z_1 z_2)^* = z_1^* \, z_2^*$;

(b) $\left(\dfrac{1}{z}\right)^* = \dfrac{1}{z^*}$;

(c) $|z_1 z_2| = |z_1| \, |z_2|$.

Problem 4. In the complex plane, describe in words and sketch the domain D given by the inequalities

$$1 < |z + \mathrm{i}| \leq 3 \; .$$

Denote the boundaries that do not belong to D with dashed lines, and the boundaries that belong to D with solid lines.

Problem 5. Evaluate $\mathrm{Re}\,\dfrac{3+\mathrm{i}}{1-\mathrm{i}}$.

Problem 6. Express the following functions in the form $w(z) = u(x,y) + \mathrm{i}\, v(x,y)$ (where u and v are real-valued functions of two real variables):

(a) $w(z) = \frac{1}{(1-z)^2}$;

(b) $\dfrac{z^*}{z}$.

Problem 7. Determine the modulus and the principal argument for the complex numbers

(a) $2 - 2\mathrm{i}$;

(b) $-\mathrm{i}$;

(c) $-3 + \sqrt{3}\,\mathrm{i}$.

Problem 8. Express the following complex numbers in Cartesian coordinates (i.e., in the form $z = x + \mathrm{i}y$):

(a) $\mathrm{e}^{\ln 2 - (\pi/4)\mathrm{i}}$;

(b) $6\,\mathrm{e}^{2\pi\mathrm{i}/3}$.

Problem 9. Evaluate $\left(\sqrt{3}+\mathrm{i}\right)^{14}$; write your result in the form $\left(\sqrt{3}+\mathrm{i}\right)^{14} = x + \mathrm{i}y$.