**MATH 3413** **Homework 2** **Due Thu, Feb 2, 2012**

**Sec. 1.6:** problems 4, 8, 22, 24, 30, 37, 38, 63, 64.

*Hint to Problem 1.6/30:* Set $u(x) = \mathrm{e}^{y(x)}$ to obtain a homogeneous equation for $u(x)$.

**Food for thought:**[1] Sec. 1.6, problems 66, 67.

[1] "Food for thought" problems are problems you should think about, but not turn in with the regular homework problems.

**Solving an exact equation.** Here we will solve the exact equation

$$\left(x^3 + \frac{y}{x}\right) dx + \left(y^2 + \ln x\right) dy = 0 \ . \tag{1}$$

First of all, let's check that it is exact:

$$M_y = \frac{\partial}{\partial y}\left(x^3 + \frac{y}{x}\right) = \frac{1}{x} \ , \qquad N_y = \frac{\partial}{\partial y}\left(y^2 + \ln x\right) = \frac{1}{x} \ ,$$

so the equation is indeed exact.

Now we will solve the equation. We are looking for a function $F(x, y)$ such that $F_x = M$ and $F_y = N$; once we find such a function, the solution will be given implicitly by the relationship $F(x, y) = 0$. From $F_x = M = \left(x^3 + \frac{y}{x}\right)$ we obtain

$$F(x, y) = \frac{1}{4}x^4 + y \ln x + g(y) \ , \tag{2}$$

where $g(y)$ is an arbitrary function of $y$. Now we use this form of $F$ to find a function $g(y)$ such that $F_y = N = (y^2 + \ln x)$. Using the particular form of $F$, we obtain

$$\frac{\partial}{\partial y}\left(\frac{1}{4}x^4 + y \ln x + g(y)\right) = y^2 + \ln x \ ,$$

or, equivalently,

$$\ln x + g'(y) = y^2 + \ln x \ . \tag{3}$$

Therefore $g'(y) = y^2$, hence $g(y) = \frac{1}{3}y^3 + C$, where $C$ is an arbitrary constant. Returning to (2), we obtain that

$$F(x, y) = \frac{1}{4}x^4 + y \ln x + \frac{1}{3}y^3 + C \ . \tag{4}$$

Therefore, the general solution of (1) is given by the relation

$$F(x, y) = \frac{1}{4}x^4 + y \ln x + \frac{1}{3}y^3 + C = 0 \ . \tag{5}$$

Note something very important: when we tried to choose a function $g(y)$ in order to satisfy $F_y = N$, we obtained the relation (3), from which we expressed $g'(y)$. The fact that equation (1) was exact guarantees that (3) will look like this:

$$g'(y) = (\text{some function of } y)\ .$$

If the right-hand side of the last relation contained $x$, then we would run into a problem because the left-hand side of this relation does not contain $x$ at all.

Let us now solve (1) by performing the operations in a different order: first we will integrate $F_y = N$, and then will adjust the result of this integration to make the function $F$ satisfy the relation $F_x = M$ as well. From $F_y = N = y^2 + \ln x$ we obtain

$$F(x,y) = \frac{1}{3}y^3 + y\ln x + f(x)\ , \tag{6}$$

where $f(x)$ is an arbitrary function of $x$. Using this particular form of $F$, we obtain

$$\frac{\partial}{\partial x}\left(\frac{1}{3}y^3 + y\ln x + f(x)\right) = y^2 + \ln x\ ,$$

that is,

$$\frac{y}{x} + f'(x) = x^3 + \frac{y}{x}\ .$$

Note that, again, there is a miraculous cancellation (due to the exactness of (1)), so that we obtain that $f'(x)$ is a function of $x$ only, namely, $f'(x) = x^3$. Integrating this, we obtain $f(x) = \frac{1}{4}x^4 + C$, i.e.,

$$F(x,y) = \frac{1}{3}y^3 + y\ln x + \frac{1}{4}x^4 + C\ ,$$

which is the same as (4).

Let's check that the function $y(x)$ defined implicitly by (5) satisfies the original equation (1). Differentiate (5) with respect to $x$, treating $y$ as a function of $x$:

$$\frac{d}{dx}\left(\frac{1}{4}x^4 + y(x)\ln x + \frac{1}{3}y(x)^3 + C\right) = 0\ .$$

Applying the product rule and the chain rule, we obtain

$$x^3 + y'(x)\ln x + \frac{y(x)}{x} + y(x)^2y'(x) = 0\ ,$$

or, equivalently,

$$x^3 + \frac{y}{x} + \left(\ln x + y^2\right)\frac{dy}{dx} = 0\ . \tag{7}$$

Finally, we note that (1) is nothing but a formal way of writing (7).

Here we would like to emphasize one very important point: the fragility of the exactness property. Let us, for example, multiply (1) by $x$ and obtain the equation

$$\big(x^4 + y\big)\,dx + \big(xy^2 + x\ln x\big)\,dy = 0 \ . \tag{8}$$

First of all, check that (8) is *not* exact:

$$\frac{\partial}{\partial y}\big(x^4 + y\big) = 1 \ , \qquad \text{while} \qquad \frac{\partial}{\partial x}\big(xy^2 + x\ln x\big) = y^2 + \ln x + 1 \ .$$

Let us ignore the fact that (8) is not exact, and try to solve it as an exact equation. We have to find a function $F(x, y)$ satisfying $F_x = x^4 + y$, $F_y = xy^2 + x\ln x$. From $F_x = x^4 + y$ we obtain

$$F(x, y) = \frac{1}{5}x^5 + xy + g(y)$$

for some arbitrary function $g(y)$. Then we try to use this form of $F(x, y)$ to satisfy $F_y = xy^2 + x\ln x$:

$$\frac{\partial}{\partial y}\left(\frac{1}{5}x^5 + xy + g(y)\right) = xy^2 + x\ln x$$

or, equivalently,

$$x + g'(y) = 2xy + \ln x + 1 \ .$$

From here we obtain

$$g'(y) = 2xy + \ln x + 1 - x \ , \tag{9}$$

which is impossible to satisfy because the right-hand side depends on $x$, while the left-hand side does not. Therefore, there is no function $g(y)$ satisfying (9), which means that (8) cannot be solved by the method for solving exact equations.