

Sec. 1.6: problems 4, 8, 22, 24, 30, 37, 38, 63, 64.

Hint to Problem 1.6/30: Set $u(x) = e^{y(x)}$ to obtain a homogeneous equation for $u(x)$.

Food for thought:¹ Sec. 1.6, problems 66, 67.

¹ “Food for thought” problems are problems you should think about, but not turn in with the regular homework problems.

Solving an exact equation. Here we will solve the exact equation

$$\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x) dy = 0 . \quad (1)$$

First of all, let's check that it is exact:

$$M_y = \frac{\partial}{\partial y} \left(x^3 + \frac{y}{x}\right) = \frac{1}{x} , \quad N_x = \frac{\partial}{\partial x} (y^2 + \ln x) = \frac{1}{x} ,$$

so the equation is indeed exact.

Now we will solve the equation. We are looking for a function $F(x, y)$ such that $F_x = M$ and $F_y = N$; once we find such a function, the solution will be given implicitly by the relationship $F(x, y) = 0$. From $F_x = M = \left(x^3 + \frac{y}{x}\right)$ we obtain

$$F(x, y) = \frac{1}{4}x^4 + y \ln x + g(y) , \quad (2)$$

where $g(y)$ is an arbitrary function of y . Now we use this form of F to find a function $g(y)$ such that $F_y = N = (y^2 + \ln x)$. Using the particular form of F , we obtain

$$\frac{\partial}{\partial y} \left(\frac{1}{4}x^4 + y \ln x + g(y)\right) = y^2 + \ln x ,$$

or, equivalently,

$$\ln x + g'(y) = y^2 + \ln x . \quad (3)$$

Therefore $g'(y) = y^2$, hence $g(y) = \frac{1}{3}y^3 + C$, where C is an arbitrary constant. Returning to (2), we obtain that

$$F(x, y) = \frac{1}{4}x^4 + y \ln x + \frac{1}{3}y^3 + C . \quad (4)$$

Therefore, the general solution of (1) is given by the relation

$$F(x, y) = \frac{1}{4}x^4 + y \ln x + \frac{1}{3}y^3 + C = 0 . \quad (5)$$

Note something very important: when we tried to choose a function $g(y)$ in order to satisfy $F_y = N$, we obtained the relation (3), from which we expressed $g'(y)$. The fact that equation (1) was exact guarantees that (3) will look like this:

$$g'(y) = (\text{some function of } y) .$$

If the right-hand side of the last relation contained x , then we would run into a problem because the left-hand side of this relation does not contain x at all.

Let us now solve (1) by performing the operations in a different order: first we will integrate $F_y = N$, and then will adjust the result of this integration to make the function F satisfy the relation $F_x = M$ as well. From $F_y = N = y^2 + \ln x$ we obtain

$$F(x, y) = \frac{1}{3}y^3 + y \ln x + f(x) , \quad (6)$$

where $f(x)$ is an arbitrary function of x . Using this particular form of F , we obtain

$$\frac{\partial}{\partial x} \left(\frac{1}{3}y^3 + y \ln x + f(x) \right) = y^2 + \ln x ,$$

that is,

$$\frac{y}{x} + f'(x) = x^3 + \frac{y}{x} .$$

Note that, again, there is a miraculous cancellation (due to the exactness of (1)), so that we obtain that $f'(x)$ is a function of x only, namely, $f'(x) = x^3$. Integrating this, we obtain $f(x) = \frac{1}{4}x^4 + C$, i.e.,

$$F(x, y) = \frac{1}{3}y^3 + y \ln x + \frac{1}{4}x^4 + C ,$$

which is the same as (4).

Let's check that the function $y(x)$ defined implicitly by (5) satisfies the original equation (1). Differentiate (5) with respect to x , treating y as a function of x :

$$\frac{d}{dx} \left(\frac{1}{4}x^4 + y(x) \ln x + \frac{1}{3}y(x)^3 + C \right) = 0 .$$

Applying the product rule and the chain rule, we obtain

$$x^3 + y'(x) \ln x + \frac{y(x)}{x} + y(x)^2 y'(x) = 0 ,$$

or, equivalently,

$$x^3 + \frac{y}{x} + (\ln x + y^2) \frac{dy}{dx} = 0 . \quad (7)$$

Finally, we note that (1) is nothing but a formal way of writing (7).

Here we would like to emphasize one very important point: the fragility of the exactness property. Let us, for example, multiply (1) by x and obtain the equation

$$(x^4 + y) dx + (xy^2 + x \ln x) dy = 0 . \quad (8)$$

First of all, check that (8) is *not* exact:

$$\frac{\partial}{\partial y} (x^4 + y) = 1 , \quad \text{while} \quad \frac{\partial}{\partial x} (xy^2 + x \ln x) = y^2 + \ln x + 1 .$$

Let us ignore the fact that (8) is not exact, and try to solve it as an exact equation. We have to find a function $F(x, y)$ satisfying $F_x = x^4 + y$, $F_y = xy^2 + x \ln x$. From $F_x = x^4 + y$ we obtain

$$F(x, y) = \frac{1}{5}x^5 + xy + g(y)$$

for some arbitrary function $g(y)$. Then we try to use this form of $F(x, y)$ to satisfy $F_y = xy^2 + x \ln x$:

$$\frac{\partial}{\partial y} \left(\frac{1}{5}x^5 + xy + g(y) \right) = xy^2 + x \ln x$$

or, equivalently,

$$x + g'(y) = 2xy + \ln x + 1 .$$

From here we obtain

$$g'(y) = 2xy + \ln x + 1 - x , \quad (9)$$

which is impossible to satisfy because the right-hand side depends on x , while the left-hand side does not. Therefore, there is no function $g(y)$ satisfying (9), which means that (8) cannot be solved by the method for solving exact equations.