MATH 5763 **Homework 11** **Due Thu, Apr 24**

Problem 1. Let $N = \{N_t : t \geq 0\}$ be a Poisson process with a constant rate λ.

(a) Compute the expected value $m_N(t) = E[N_t]$.

Hint: Use that N_t is a Poisson distributed random variable with parameter λt. Recall that if $X \sim \text{Poisson}(\alpha)$, then $E[X] = \alpha$ and $\operatorname{Var} X = \alpha$.

(b) Use your result in (a) and the basic properties of Poisson processes to find the autocorrelation function, $R_N(s,t) = E[N_s N_t]$, of the Poisson process N.

Hint: First assume that $0 \leq x \leq t$ and compute $R_N(s,t)$; then think how your result will change if $0 \leq t \leq s$, and write your result for $R_N(s,t)$ in the general case.

(c) Compute the autocovariance function,

$$C_N(s,t) = E[(N_s - EN_s)(N_t - EN_t)] = R_N(s,t) - m_N(s)m_N(t) ,$$

and the correlation coefficient, $\rho_N(s,t) = \dfrac{C_N(s,t)}{\sqrt{C_N(s,s)\, C_N(t,t)}}$.

(d) Look at $C_N(t,t)$ and at the hint to part (a) – are the results for $C_N(t,t)$ and $\operatorname{Var} X_t$ consistent?

(e) Is N a wide-sense stationary process? Explain briefly.

(f) Is N a strong-sense stationary process? Why?

(g) Recall that the *probability mass function of order* k of a discrete state space stochastic process X was defined (on page 49 of the book) as

$$p(x_1, \ldots, x_k; t_1, \ldots, t_k) = P(X_{t_1} = x_1, \ldots, X_{t_k} = x_k) .$$

Determine the probability mass functions of orders 1, 2, and 3, $p(i;r)$, $p(i,j;r,s)$, $p(i,j,k;r,s,t)$, of the Poisson process N assuming that $0 \leq r \leq s \leq t$, and that i, j, k are natural numbers. Clearly, since the Poisson process is non-decreasing, $p(i,j;r,s)$ and $p(i,j,k;r,s,t)$ will be non-zero only if $i \leq j \leq k$.

Problem 2. Let $X = \{X_t : t \geq 0\}$ be a time-homogeneous continuous-time Markov process with state space $\mathcal{X} = \eta\mathbb{Z}$; $\eta\mathbb{Z}$ is a shorthand notation for the set of all integer multiples of η:

$$X_t : \Omega \to \mathcal{X} = \eta\mathbb{Z} := \{\ldots, -2\eta, -\eta, 0, \eta, 2\eta, \ldots\} .$$

The process X is allowed to jump "up" or "down" by η with equal probabilities (like in the case of a symmetric simple random walk). Let the intensity of the process X be τ, i.e.,

$$p_{jk}(h) := P\big(X_{t+h} = k\eta \,\big|\, X_t = j\eta\big) = \begin{cases} \tau h + o(h) & \text{for } k = j \pm 1 , \\ 1 - 2\tau h + o(h) & \text{for } k = j , \\ o(h) & \text{otherwise} . \end{cases}$$

(a) Show that the probabilities $p_k(t) := P(X_t = k\eta)$ satisfy the system of ODEs

$$p_k'(t) = \tau\left[p_{k-1}(t) - 2p_k(t) + p_{k+1}(t)\right] .$$

(b) Use the system of ODEs for the probabilities $p_k(t)$ to show that the characteristic function

$$\phi(\xi, t) = E\left[\mathrm{e}^{\mathrm{i}\xi X_t}\right] = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\xi k\eta}\, p_k(t)$$

satisfies the equation $\dfrac{\partial \phi}{\partial t} = \tau\left(\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta}\right)\phi$.

(c) Assume that at $t = 0$, the process was at 0 (i.e., $X_0 = 0$). What does this imply for $\phi(\xi, 0)$? Solve the equation for $\phi(\xi, t)$ derived in part (b) with the initial condition you just found.

Hint: Although the equation for $\phi(\xi, t)$ derived in (b) is about a function of two variables (namely, ξ and t), it does not contain ξ-derivatives, so you can solve it simply as an ordinary differential equation treating ξ as a fixed number. The ODE you then have to solve is of the simplest kind, $x'(t) = \alpha x(t)$, where $\alpha = \text{const}$.

(d) Now let the "spatial step-size" η of the process go to zero, and the "temporal intensity" τ of the process go to infinity, in such a way that $2\eta^2\tau \to 1$. Compare the expression for the characteristic function $\phi(\xi, t)$ in this limit with the characteristic function $\phi_{\mathrm{N}(\mu,\sigma^2)}(\xi) = \mathrm{e}^{\mathrm{i}\mu\xi - \frac{1}{2}\sigma^2\xi^2}$ of a normal random variable with mean μ and variance σ^2. What can you conclude about the distribution of the random variable X_t in the limit $\eta \to 0$, $\tau \to \infty$, $2\eta^2\tau \to 1$?

Hint: To performed the limiting transition, you can expand the expression $\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta}$ (which will be part of your result for $\phi(\xi, t)$) in a Taylor series with respect to η around the point $\eta = 0$, and, after the obvious cancellations, you will obtain

$$\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta} = -\eta^2\xi^2 + o(\xi^2) .$$

Problem 3. Let B_t be a standard Wiener process, i.e., a process with independent increments and continuous sample paths satisfying the initial condition $B_0 = 0$, and such that $B_{s+t} - B_s$ is $\mathrm{N}(0, (\sqrt{t})^2)$ for all $s \geq 0$ and $t > 0$. Let α be a positive constant. Show that $\alpha B_{t/\alpha^2}$ is a standard Wiener process. In other words, it is enough to show that the process $V_t := \alpha B_{t/\alpha^2}$ satisfies:

- $V_0 = 0$ with probability 1;
- has independent increments (i.e., if $0 \leq t_1 < t_2 < t_3 < t_4$, then $V_{t_2} - V_{t_1}$ and $V_{t_4} - V_{t_3}$ are independent random variables);
- has stationary increments (i.e., $V_{s+t} - V_s$ does not depend on s);
- $V_t \sim N(0, (\sqrt{t})^2)$.

Remark: One can also prove that $B_{s+t} - B_s$ (for any $s \geq 0$) and $tB_{1/t}$ are standard Wiener processes. (You don't need to do this here.)

Problem 4. Let N be a Poisson process with intensity λ. Since N is a Markov process, for $0 \le s \le t$, the conditional expectation of N_t conditioned on the whole history before the moment s (namely, on the knowledge of $\{N_r : 0 \le r \le s\}$) is the same as the conditional expectation of N_t conditioned on the value of N_s:

$$E[N_t | N_r : 0 \le r \le s] = E[N_t | N_s]$$

(see Definition 2.4.3 on page 61 of the book).

(a) Compute $E[N_t | N_r : 0 \le r \le s] = E[N_t | N_s]$.

Hint: Look at of Problem 3 of Homework 6 and Problem 2 of Homework 10.

(b) A continuous-time Markov process X is said to be a martingale if $E[X_t | X_r : 0 \le r \le s] = X_s$. If the process X is Markov, then the condition for being a martingale becomes simply $E[X_t | X_s] = X_s$.

Is the process N a martingale? Explain briefly.

(c) Define the stochastic process $M = \{M_t : t \ge 0\}$ by $M_t = N_t - \lambda t$. Is the process M a martingale? Explain.

Problem 5. Use the definition of the Dirac δ-function and its derivatives to compute the integrals

$$\int_{\mathbb{R}} x^{17}\, \delta(x)\, \mathrm{d}x \ , \qquad \int_{\mathbb{R}} \mathrm{e}^{5x^2}\, \delta(x)\, \mathrm{d}x \ , \qquad \int_{\mathbb{R}} \mathrm{e}^{5x^2}\, \delta'(x)\, \mathrm{d}x \ , \qquad \int_{\mathbb{R}} \mathrm{e}^{5x^2}\, \delta'(x-3)\, \mathrm{d}x \ .$$